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SEMI-PARAMETRIC UPPER BOUNDS FOR OPTION PRICES AND EXPECTED PAYOFFS*

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Upper bounds on the expected payoff of call and put options are derived. These bounds depend only on the mean and variance of the terminal stock price and not on its entire distribution, so they are termed semi-parametric. A corollary of this result is a set of upper bounds for option prices obtained by the risk-neutral valuation approach of Cox and Ross. As an example, these bounds are shown to obtain across both lognormal diffusions and mixed diffusion-jump processes for any given data set. We present an illustrative example that suggests these bounds may be of considerable practical value.

1. Introduction

In this paper, I derive upper bounds for the expected payoff of call and put options. These upper bounds possess a remarkable property: they depend only on the mean and variance of the stock price at maturity, and not on its entire distribution. I also show how to derive corresponding semi-parametric upper bounds for option prices whenever the option's risk-neutral pricing distribution is known [see Cox and Ross (1976) and Harrison and Kreps (1979) for a formal definition of the risk-neutral pricing distribution]. Specifically, if the option's price is an appropriately discounted expected value of its terminal payoff, an upper bound for its price can be obtained by discounting the upper bound of its expected payoff. In such instances, my results complement those of Perrakis and Ryan (1984), in which bounds are obtained by essentially deriving restrictions on the discount factor (and not the expected payoff).[1]

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[1] For example, Perrakis and Ryan (1984) develop option-pricing bounds in discrete time under Rubinstein's (1976) assumptions of competitive and Pareto-efficient financial markets. Although their bounds obtain for stock price distributions with a 'non-negative beta' property, the actual values of the bounds depend critically on the specific stock price distribution assumed and are

Also, because the upper bounds proposed in this paper are shown to obtain *across* distinct families of stock price distributions, they may be viewed as a partial complement to Merton's (1972) non-parametric lower bounds.

In section 2 the upper bound for expected payoffs of call and put options is presented. I show in section 3 how to obtain upper bounds on the option's price under certain conditions. Section 4 provides an important illustrative example, and my conclusions are presented in section 5.

2. An upper bound for an option's expected payoff

Consider a call option at time t with strike price K on a stock with price $S(t)$ expiring at date T. The date-t expectation X_t of its terminal payoff is simply given by $X_t \equiv \mathrm{E}_t[\max[S(T)-K,0]]$, where $\mathrm{E}_t[\cdot]$ denotes the conditional expectations operator. Assume that the conditional mean $\mu(t)$ and variance $\sigma^2(t)$ are both finite for any t.[2] The distribution of $S(T)$ is otherwise allowed to be completely arbitrary. The following proposition presents an upper bound for X_t over all possible distributions of the terminal stock price $S(T)$ with mean $\mu(t)$ and variance $\sigma^2(t)$.

Proposition. *Let F_t denote the set of all conditional distributions of $S(T)$ with mean $\mu(t)$ and variance $\sigma^2(t)$. Let $\overline{X}_t$ be defined as the upper bound of X_t over the set F_t, i.e.,*

$$\overline{X}_t \equiv \max_{F_t \in F_t} X_t \equiv \max_{F_t \in F_t} \left\{ \int \max[S(T)-K,0] \, \mathrm{d}F_t(S(T)) \right\}. \tag{1}$$

Then the solution to (1) exists and is given by

$$\begin{aligned} \overline{X}_t &= \frac{\mu(t)\cdot[\mu(t)-K]+\mu(t)\sigma^2(t)}{\mu(t)^2+\sigma(t)^2} && \text{if} \quad K \le \frac{\mu^2(t)+\sigma^2(t)}{2\mu(t)}, \\ &= \tfrac{1}{2}\left[\mu(t)-K+\sqrt{[K-\mu(t)]^2+\sigma^2(t)}\right] && \text{if} \quad K > \frac{\mu^2(t)+\sigma^2(t)}{2\mu(t)}. \end{aligned} \tag{2}$$

therefore parametric. Levy (1985) also considers discrete-time option-pricing bounds for non-negative beta distributions but uses a stochastic dominance approach. His bounds are also parametric, since their particular values depend on the distribution function of the stock price. Ritchken (1985) uses a linear-programming approach to obtain bounds that also require parametric assumptions (state probabilities) to calculate numerical values. Finally Perrakis (1986) extends the results of Perrakis and Ryan (1984) by tightening the bounds under further restrictions on the stock price distribution (finite upper limit, positive lower limit) and by considering bounds on American put options. These are also parametric in nature.

[2] Of course, this does *not* rule out the possibility of a time-varying volatility that has been shown to generate leptokurtotic distributions for stock returns.

Proof. See appendix.

Note that, other than fixing the first two moments, I have placed no restrictions on $F_t(S(T))$; it may be a continuous distribution, discrete, or mixed. Moreover, whether the process $S(T)$ is defined in discrete or continuous time is irrelevant. The proposition requires only that the conditional distribution of $S(T)$ has finite mean and variance.

Using this proposition, the upper bound $\overline{Y}_t$ for the expected payoff of a put option with strike price K on the same stock follows directly since

$$\max[S(T)-K,0]=\max[K-S(T),0]+S(T)-K. \tag{3}$$

Taking the expectation of both sides of (3) and using the definition of $\overline{X}_t$ and $\overline{Y}_t$ then yields:

Corollary. The semi-parametric upper bound $\overline{Y}_t$ for the expected payoff of a put option is given by

$$\overline{Y}_t=\overline{X}_t-\mu(t)+K. \tag{4}$$

3. Upper bounds for option prices

When the appropriate risk-neutral pricing distribution is known, an option's price can be expressed as its expected payoff discounted at the risk-free rate where the expectation is computed using its risk-neutral distribution. A semi-parametric upper bound for the price of an option can therefore be obtained in all cases where the risk-neutral distribution is known. Since Cox and Ross (1976) and Harrison and Kreps (1979) have shown the risk-neutral valuation technique to be quite general indeed, it would then seem that semi-parametric upper bounds for option prices must also generally exist. Although existence is indeed insured, care must be taken in interpreting the upper bound empirically. This point is perhaps best understood through the following simple example.[3] Consider a two-date two-state economy in which markets are complete. Denote the dates 0 and 1 and the (date-1) states a and b, respectively. Market completeness implies the existence of two pure Arrow–Debreu contingent claims with date-0 prices p_a and p_b, where the former security pays \$1 in state a and 0 in state b, and vice versa for the latter security. In this economy, the date-0 price C_0 of a call option with strike price

[3] I am grateful to the referee for suggesting this example.

K on a stock with date-1 price $S_1 = \{S_a \text{ or } S_b\}$ is simply given by

$$\begin{aligned} C_0 &= p_a \cdot \max[S_a - K, 0] + p_b \cdot \max[S_b - K, 0] \\ &= \left[\frac{p_a}{p_a + p_b} \cdot \max[S_a - K, 0] + \frac{p_a}{p_a + p_b} \cdot \max[S_b - K, 0] \right] \\ &\quad \cdot (p_a + p_b) \\ &= [p_a^* \cdot \max[S_a - K, 0] + p_b^* \cdot \max[S_b - K, 0]] \cdot (1 + R_f)^{-1}, \end{aligned} \tag{5}$$

where R_f is the rate of return on the riskless asset. Note that p_a^* and p_b^* are both non-negative and sum to one, hence they may be interpreted as probabilities. Therefore, we obtain the well-known risk-neutral valuation result that the option price is equal to the expected payoff discounted at the riskless rate, where the expectation is taken with respect to the risk-neutral distribution (the p_i^*'s), i.e.,

$$C_0 = (1 + R_f)^{-1} \cdot \mathrm{E}^*[\max[S_1 - K, 0]]. \tag{6}$$

In the terminology of Harrison and Kreps (1979), the probabilities p_a^* and p_b^* form the equivalent martingale measure under which all assets must earn the riskless rate of return. Of course, prices in almost any equilibrium model may be re-expressed as an expectation with respect to a risk-neutral distribution. It is only when markets are complete, however, that this distribution is unique.[4]

By applying the proposition to (6), we may readily obtain a semi-parametric upper bound $\overline{C}_0$ for C_0. But observe that, according to our proposition, $\overline{C}_0$ is an upper bound for C_0 over the class of p_i^*'s that have the same variance V^* [they all have the same mean $S_0(1 + R_f)$ by construction]. Herein lies the difficulty: the upper bound obtains over the set of *risk-neutral* distributions for S_1 of a given variance, not over the set of *actual* distributions for S_1. Therefore, we must interpret $\overline{C}_0$ as an upper bound over all terminal stock price distributions whose corresponding risk-neutral distributions have common variance V^*. To determine precisely the class of actual stock price distributions for which this upper bound is binding, we must be able to relate the set of risk-neutral distributions of a common variance V^* to a set of empirically observable distributions. Such a relation is generally quite difficult to derive. Even if markets are complete (as in the example above) so that the risk-neutral distribution is unique, the relation between the actual and risk-

[4] See Harrison and Kreps (1979). See Jones (1984) for pricing options not spanned by stock and bonds alone.

neutral distributions of the terminal stock price will still depend on the state prices and, consequently, preferences and endowments.

That the upper bound may depend on state prices is not surprising. After all, the bounds of Perrakis and Ryan (1984) and Perrakis (1986) also require restricting the preferences of agents in an equilibrium pricing model.[5] Now given a specific equilibrium asset-pricing framework, the relation between risk-neutral and actual distributions may often be derived explicitly. However, it is the lack of any *general* relation between the two distributions that prevents the semi-parametric upper bounds from having wider empirical relevance. Nevertheless, we show in the next section that these bounds are still of considerable interest when there are several competing specifications for the stochastic process of the stock price.

4. Calculating the bounds for selected stochastic processes

In this section, we illustrate the practical relevance of the semi-parametric upper bounds by relating the risk-neutral distributions of two specific stock price processes to observable quantities. In particular, we show that the estimated upper bounds will be the same irrespective of whether the stock price follows a lognormal diffusion or Merton's (1976a) mixed diffusion-jump process.[6]

Let $C(S(t), t)$ and $P(S(t), t)$ denote the date-t price of European call and put options respectively with time to maturity $\tau \equiv T - t$ and strike price K, and suppose that both options may be priced via the risk-neutral distribution $F_t^*(S(T))$ so that

$$C(S(t), t) = \mathrm{e}^{-r\tau} \int_0^{\infty} \max[S(T) - K, 0]\, \mathrm{d}F_t^*(S(T)), \tag{7a}$$

$$P(S(t), t) = \mathrm{e}^{-r\tau} \int_0^{\infty} \max[K - S(T), 0]\, \mathrm{d}F_t^*(S(T)). \tag{7b}$$

Now define the variance of the risk-neutral distribution as V^*, where[7]

$$V^* \equiv \int \left[\frac{S(T)}{S(t)} - \mathrm{e}^{r\tau} \right]^2 \mathrm{d}F_t^*(S(T)). \tag{8}$$

[5] Specifically, using Rubinstein's (1976) framework, they require that the normalized conditional expected marginal utility of consumption is non-increasing in the price change.

[6] Of course, the choice between the two stochastic processes must ultimately be motivated by economic considerations. For example, the particular source of variability in observed prices plays a critical role in determining an appropriate stochastic specification; volatility arising from bid–ask spreads is unlikely to have the same impact on option prices as variance from other forces [see, for example, Phillips and Smith (1980) and Amihud and Mendelson (1986)].

[7] Note that I have defined the variance of the gross return $S(T)/S(t)$ as V^*, not the variance of the terminal stock price $S(T)$. This is merely for computational convenience in the formulas that follow.

A straightforward application of the proposition of section 2 then yields semi-parametric upper bounds $\bar{C}(t)$ and $\bar{P}(t)$ for the option prices, where these bounds obtain over all risk-neutral distributions with variance V^*,

$$\bar{C}(t) = \frac{S(t) - K\,\mathrm{e}^{-r\tau} + S(t)V^*\mathrm{e}^{-2r\tau}}{1 + V^*\mathrm{e}^{-2r\tau}} \quad \text{if} \quad \frac{S(t)}{K} \geq \frac{2\,\mathrm{e}^{-r\tau}}{1 + V^*\mathrm{e}^{-2r\tau}},$$

$$= \tfrac{1}{2}\left[S(t) - K\,\mathrm{e}^{-r\tau} + \sqrt{(K\,\mathrm{e}^{-r\tau} - S(t))^2 + S^2(t)V^*\mathrm{e}^{-2r\tau}}\right] \tag{9a}$$

$$\text{if} \quad \frac{S(t)}{K} < \frac{2\,\mathrm{e}^{-r\tau}}{1 + V^*\mathrm{e}^{-2r\tau}},$$

$$\bar{P}(t) = \bar{C}(t) - S(t) + K\,\mathrm{e}^{-r\tau}. \tag{9b}$$

It can readily be shown that these bounds satisfy many of the well-known properties of rationally determined option prices [e.g., increasing in r, τ, and V^*; homogeneity of degree one in S and K; etc.].

To interpret (9), we must relate the risk-neutral variance V^*, which is usually unobservable, to observable quantities. To do this, suppose we have two candidates for the stochastic process driving $S(t)$, which we denote by $S_1(t)$ and $S_2(t)$. Specifically let S_1 and S_2 satisfy the following stochastic differential equations:

$$\mathrm{d}S_1 = \alpha_1 S_1\,\mathrm{d}t + \sigma_1 S_1\,\mathrm{d}W, \tag{10a}$$

$$\mathrm{d}S_2 = [\alpha_2 - \lambda(k-1)]S_2\,\mathrm{d}t + \sigma_2 S_2\,\mathrm{d}W + (\tilde{\gamma} - 1)S_2\,\mathrm{d}N_\lambda, \tag{10b}$$

where

$$\ln\tilde{\gamma} \sim \mathrm{N}(\beta, \delta^2) \quad \text{and} \quad k \equiv \mathrm{E}[\tilde{\gamma}] = \mathrm{e}^{(\beta + \frac{1}{2}\delta^2)}.$$

S_1 is the well-known lognormal diffusion, and S_2 is Merton's (1976a) mixed diffusion-jump process. It can be shown that to value options on S_1 and S_2 via risk-neutral arguments requires the adjustments $\alpha_1 = r$ and $\alpha_2 = r$, and the additional assumption that the jump risk is completely diversifiable. Using the adjusted or risk-neutral distributions of $S_1(T)$ and $S_2(T)$ to compute the expected payoff and then discounting at the riskless rate then yields the option prices. A simple calculation yields the variances of the risk-neutral distributions as

$$V_1^* = e^{2r\tau} \cdot \left[e^{\sigma_1^2 \tau} - 1\right] \equiv e^{2r\tau} \cdot \theta_1, \tag{11a}$$

$$V_2^* = e^{2r\tau} \cdot \left[e^{[\lambda(k-1)^2 + \sigma_2^2 + \lambda\sigma_\gamma^2]\tau} - 1\right] \equiv e^{2r\tau} \cdot \theta_2, \tag{11b}$$

where

$$\sigma_\gamma^2 \equiv \operatorname{var}[\tilde{\gamma}] = e^{[2\beta+\delta^2]} \cdot \left(e^{[\delta^2]} - 1\right).$$

In principle the parameters of the right-hand sides of eqs. (11) can be estimated. Moreover, in this example it can be shown that the method-of-moments estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ_1 and θ_2 are numerically identical. Hence for a given set of observations on the stock price process, the estimated risk-neutral variance of the lognormal diffusion $\hat{V}_1^*$ is the same as that of the mixed diffusion-jump process $\hat{V}_2^*$. Since $\hat{V}_1^* = \hat{V}_2^*$, the estimated semi-parametric upper bound will also be identical for these two stochastic processes, so that the bound must obtain across *both* the set of lognormal diffusions and mixed diffusion-jump processes.[8]

Tables 1a–1c give the upper bounds $\bar{C}$ and $\bar{P}$ for call and put options on a stock (with current price \$40) for a variety of strike-price/time-to-maturity/variance combinations. For purposes of comparison, theoretical Black–Scholes and Merton mixed diffusion-jump process prices are also computed. As a matter of convention only, the parameter σ_1^2 is implicitly defined by the definitional relation $s \equiv 52\sigma_1^2$, where s (labeled 'standard deviation') ranges from 0.20 to 0.80. If the true stock price process were a lognormal diffusion, this range would imply annual standard deviations of 20% to 80% for continuously compounded returns. Parameters for the mixed

[8]In this example the equality of $\hat{V}_1^*$ with $\hat{V}_2^*$ does not hinge specifically on the use of method-of-moments estimators, but will obtain for any set of parameter estimators that imply the same sample mean and variance estimates. That is, $\hat{\alpha}_1 = \hat{\alpha}_2$ and $\hat{V}_1 = \hat{V}_2$ implies $\hat{V}_1^* = \hat{V}_2^*$. But the equalities are quite reasonable, since they simply require that the estimated parameters of the two processes be such that they yield the same estimated mean and variance of the stock's return. Since the two processes are competing specifications of the same data set, one would hope that the estimated moments implied by the various parameter estimates are numerically close (and close to the sample moments of the data). In fact, Hansen's (1982) and Hansen and Singleton's (1982) generalization of the method-of-moments estimation technique (which also yields $\hat{\alpha}_1 = \hat{\alpha}_2$ and $\hat{V}_1 = \hat{V}_2$) contains many of the most popular estimators as special cases.

Table 1a

Comparison of Black–Scholes and Merton call and put option prices C_{BS}, C_M, P_{BS}, P_M with corresponding semi-parametric upper bounds $\bar{C}$ and $\bar{P}$ for options with strike price $K = 30, 35, 40, 45, 50$ and time to maturity $\tau = 1$ week on a stock with a current price of \$40 and (annual) compound standard deviations $s = 0.2, 0.4, 0.6, 0.8$, given a 6% annual riskless rate of interest. The σ_1 parameter of the lognormal diffusion used in computing C_{BS} and P_{BS} is calculated as $\sigma_1 \equiv s/\sqrt{52}$. Parameters of the mixed diffusion-jump used in computing C_M and P_M are given by $k \equiv 1$, $\lambda \equiv 0.25$, $\sigma_2^2 \equiv 0.10 \cdot \sigma_1^2$, and $\mathrm{var}[\gamma] \equiv 3.60 \cdot \sigma_1^2$. Note that $V_1^* \equiv e^{2r\tau} \cdot (e^{\sigma_1^2 \tau} - 1) = V_2^*$.

Time to maturity $\tau = 1$ week

Std. dev. s	K	Calls C_{BS}	C_M	$\bar{C}$	Puts P_{BS}	P_M	$\bar{P}$	Max. upper bound minus price[a]
				$V_1^* = 7.7125 \times 10^{-4}$				
0.20	30	10.034	10.034	10.064	0.000	0.000	0.031	0.031
0.20	35	5.039	5.041	5.100	0.000	0.002	0.060	0.060
0.20	40	0.465	0.330	0.578	0.420	0.285	0.533	0.248
0.20	45	0.000	0.005	0.061	4.950	4.954	5.011	0.061
0.20	50	0.000	0.000	0.031	9.944	9.944	9.975	0.031
				$V_1^* = 3.0886 \times 10^{-3}$				
0.40	30	10.034	10.036	10.155	0.000	0.002	0.121	0.121
0.40	35	5.044	5.093	5.273	0.005	0.054	0.234	0.229
0.40	40	0.907	0.635	1.133	0.862	0.591	1.088	0.498
0.40	45	0.015	0.083	0.238	4.965	5.032	5.187	0.223
0.40	50	0.000	0.012	0.122	9.944	9.956	10.066	0.122
				$V_1^* = 6.9627 \times 10^{-3}$				
0.60	30	10.034	10.059	10.303	0.000	0.025	0.270	0.270
0.60	35	5.108	5.206	5.541	0.069	0.166	0.502	0.432
0.60	40	1.349	0.939	1.690	1.304	0.894	1.645	0.751
0.60	45	0.129	0.226	0.509	5.078	5.176	5.459	0.380
0.60	50	0.004	0.075	0.272	9.948	10.019	10.216	0.268
				$V_1^* = 1.2412 \times 10^{-2}$				
0.80	30	10.039	10.114	10.505	0.006	0.081	0.472	0.466
0.80	35	5.265	5.346	5.881	0.226	0.307	0.842	0.616
0.80	40	1.791	1.240	2.248	1.746	1.195	2.203	1.008
0.80	45	0.355	0.396	0.854	5.304	5.346	5.803	0.499
0.80	50	0.042	0.186	0.475	9.986	10.130	10.419	0.434

[a] This column reports the larger of the two differences $\bar{C} - C_{BS}$ and $\bar{C} - C_M$. The difference between the Black–Scholes/Merton call prices and their upper bound is numerically identical to the difference between put prices and their upper bounds ceteris paribus because of the put–call parity relation. Since all numerical calculations were performed in double precision, the numbers reported in the last column may not match (to the third decimal place) the difference of the reported numbers because of rounding.

Table 1b

Comparison of Black–Scholes and Merton call and put option prices C_{BS}, C_M, P_{BS}, P_M with corresponding semi-parametric upper bounds $\bar{C}$ and $\bar{P}$ for options with strike price $K = 30, 35, 40, 45, 50$ and time to maturity $\tau = 12$ weeks on a stock with a current price of \$40 and (annual) compound standard deviations $s = 0.2, 0.4, 0.6, 0.8$, given a 6% annual riskless rate of interest. The σ_1 parameter of the lognormal diffusion used in computing C_{BS} and P_{BS} is calculated as $\sigma_1 \equiv s/\sqrt{52}$. Parameters of the mixed diffusion-jump used in computing C_M and P_M are given by $k \equiv 1$, $\lambda \equiv 0.25$, $\sigma_2^2 \equiv 0.10 \cdot \sigma_1^2$, and $\mathrm{var}[\gamma] \equiv 3.60 \cdot \sigma_1^2$. Note that $V_1^* \equiv e^{2r\tau} \cdot (e^{\sigma_1^2\tau} - 1) = V_2^*$.

Time to maturity $\tau = 12$ weeks

Std. dev. s	K	Calls C_{BS}	C_M	$\bar{C}$	Puts P_{BS}	P_M	$\bar{P}$	Max. upper bound minus price[a]
				$V_1^* = 9.5263 \times 10^{-3}$				
0.20	30	10.401	10.405	10.746	0.001	0.004	0.345	0.344
0.20	35	5.565	5.584	6.078	0.098	0.117	0.610	0.513
0.20	40	1.804	1.747	2.212	1.270	1.213	1.677	0.464
0.20	45	0.285	0.295	0.724	4.684	4.694	5.123	0.439
0.20	50	0.022	0.039	0.382	9.354	9.371	9.714	0.360
				$V_1^* = 3.8639 \times 10^{-2}$				
0.40	30	10.566	10.600	11.688	0.165	0.200	1.287	1.122
0.40	35	6.378	6.353	7.479	0.910	0.885	2.012	1.126
0.40	40	3.316	3.193	4.155	2.781	2.658	3.621	0.962
0.40	45	1.493	1.426	2.260	5.892	5.825	6.659	0.834
0.40	50	0.594	0.611	1.402	9.926	9.943	10.734	0.808
				$V_1^* = 8.8987 \times 10^{-2}$				
0.60	30	11.154	11.160	13.055	0.753	0.760	2.654	1.901
0.60	35	7.543	7.434	9.224	2.075	1.966	3.757	1.790
0.60	40	4.825	4.626	6.160	4.291	4.091	5.625	1.534
0.60	45	2.947	2.784	4.084	7.346	7.183	8.483	1.300
0.60	50	1.735	1.668	2.845	11.067	11.001	12.178	1.177
				$V_1^* = 1.6349 \times 10^{-1}$				
0.80	30	12.019	11.950	14.724	1.619	1.549	4.324	2.775
0.80	35	8.817	8.610	11.168	3.349	3.142	5.700	2.558
0.80	40	6.325	6.034	8.250	5.791	5.500	7.716	2.216
0.80	45	4.463	4.194	6.077	8.862	8.593	10.476	1.883
0.80	50	3.112	2.928	4.577	12.444	12.261	13.909	1.649

[a] This column reports the larger of the two differences $\bar{C} - C_{BS}$ and $\bar{C} - C_M$. The difference between the Black–Scholes/Merton call prices and their upper bound is numerically identical to the difference between put prices and their upper bounds ceteris paribus because of the put–call parity relation. Since all numerical calculations were performed in double precision, the numbers reported in the last column may not match (to the third decimal place) the difference of the reported numbers because of rounding.

Table 1c

Comparison of Black–Scholes and Merton call and put option prices C_{BS}, C_M, P_{BS}, P_M with corresponding semi-parametric upper bounds $\bar{C}$ and $\bar{P}$ for options with strike price $K = 30, 35, 40, 45, 50$ and time to maturity $\tau = 24$ weeks on a stock with a current price of \$40 and (annual) compound standard deviations $s = 0.2, 0.4, 0.6, 0.8$, given a 6% annual riskless rate of interest. The σ_1 parameter of the lognormal diffusion used in computing C_{BS} and P_{BS} is calculated as $\sigma_1 \equiv s/\sqrt{52}$. Parameters of the mixed diffusion-jump used in computing C_M and P_M are given by $k \equiv 1$, $\lambda \equiv 0.25$, $\sigma_2^2 \equiv 0.10 \cdot \sigma_1^2$, and $\operatorname{var}[\gamma] \equiv 3.60 \cdot \sigma_1^2$. Note that $V_1^* \equiv e^{2r\tau} \cdot (e^{\sigma_1^2\tau} - 1) = V_2^*$.

Time to maturity $\tau = 24$ weeks								
Std. dev. s	K	Calls C_{BS}	C_M	$\bar{C}$	Puts P_{BS}	P_M	$\bar{P}$	Max. upper bound minus price[a]
				$V_1^* = 1.9663 \times 10^{-2}$				
0.20	30	10.812	10.821	11.447	0.016	0.025	0.651	0.635
0.20	35	6.221	6.228	6.994	0.292	0.300	1.066	0.773
0.20	40	2.710	2.673	3.312	1.649	1.612	2.250	0.639
0.20	45	0.855	0.838	1.425	4.661	4.644	5.231	0.586
0.20	50	0.199	0.214	0.788	8.872	8.887	9.461	0.589
				$V_1^* = 8.0876 \times 10^{-2}$				
0.40	30	11.361	11.369	13.131	0.565	0.573	2.335	1.770
0.40	35	7.639	7.595	9.245	1.710	1.666	3.316	1.650
0.40	40	4.817	4.731	6.093	3.756	3.670	5.032	1.362
0.40	45	2.875	2.798	3.952	6.681	6.604	7.758	1.154
0.40	50	1.641	1.605	2.696	10.314	10.278	11.370	1.091
				$V_1^* = 1.9074 \times 10^{-1}$				
0.60	30	12.535	12.487	15.470	1.739	1.691	4.674	2.983
0.60	35	9.395	9.283	11.969	3.467	3.355	6.041	2.686
0.60	40	6.919	6.769	9.050	5.857	5.707	7.989	2.282
0.60	45	5.029	4.881	6.810	8.835	8.687	10.616	1.930
0.60	50	3.623	3.505	5.208	12.297	12.178	13.882	1.703
				$V_1^* = 3.6263 \times 10^{-1}$				
0.80	30	13.976	13.847	18.305	3.180	3.050	7.509	4.459
0.80	35	11.231	11.034	15.058	5.302	5.105	9.129	4.024
0.80	40	8.994	8.759	12.267	7.933	7.698	11.206	3.508
0.80	45	7.194	6.953	9.975	11.000	10.758	13.781	3.022
0.80	50	5.755	5.533	8.164	14.428	14.206	16.837	2.631

[a] This column reports the larger of the two differences $\bar{C} - C_{BS}$ and $\bar{C} - C_M$. The difference between the Black–Scholes/Merton call prices and their upper bound is numerically identical to the difference between put prices and their upper bounds ceteris paribus because of the put–call parity relation. Since all numerical calculations were performed in double precision, the numbers reported in the last column may not match (to the third decimal place) the difference of the reported numbers because of rounding.

diffusion-jump process are chosen on the basis of the following restrictions:

$$k = 1.0, \tag{12a}$$

$$\lambda = 0.25, \tag{12b}$$

$$\theta_1 = \theta_2, \tag{12c}$$

$$\lambda\sigma_Y^2/\left(\sigma_2^2 + \lambda\sigma_Y^2\right) = 0.90. \tag{12d}$$

Restriction (12a) implies that on average the jump component will have no effect on the stock price, i.e., the jump component contains no trend. Since we will choose (arbitrarily) to measure time in weeks, restriction (12b) implies that, on average, there will be one jump every four weeks. Condition (12c) has already been discussed; it is an implication of virtually any reasonable estimation method. Although this condition will have the effect of changing the total variance of both processes in tandem as σ_1 varies between 0.20 and 0.80, restriction (12d) insures that the fraction of the total 'instantaneous' variance due to the jump component in process S_2 is always 90%. In summary, the diffusion-jump process satisfying these four restrictions will exhibit monthly jumps on average with no expected gain or loss and will possess the same volatility as its lognormal counterpart, with most of that volatility arising from the jump component.

Table 1a shows that the semi-parametric upper bound for a call with strike price \$35 and one week to go is \$5.10 when $s = 0.20$. This upper bound obtains across all stock price processes that have a risk-neutral variance V^* of 7.7125×10^{-4} over a one-week period. Assuming that this stock has an expected return of 15%, its variance under a lognormal diffusion when $s = 0.20$ is given by $V_1 = 7.7367 \times 10^{-4}$. Therefore, the upper bound \$5.10 obtains over all lognormal diffusions and mixed diffusion-jump processes that have gross return variances equal to (or less than) 7.7367×10^{-4}. Moreover, this bound obtains irrespective of whether the economy is in equilibrium or not.[9]

Several other aspects of table 1a are worth noting. In particular, observe that with one week to maturity the maximum difference between the upper bound and the option prices is generally quite small. The largest difference under an annual standard deviation of 40% is only 50 cents! This is somewhat surprising in light of Merton's (1976b) finding that in the presence of jumps the percentage error of the Black–Scholes formula may be enormous for deep-out-of-the-money options. In fact, the largest difference between the deep-out-of-the-money ($K = 50$) Black–Scholes/Merton price and its semi-

[9]Of course, uniqueness of the risk-neutral distribution does relate to market equilibrium as discussed in Harrison and Kreps (1979).

parametric upper bound with one week to go is only 43 cents, and that is only for an extreme variance parameter. Since the upper bound obtains across lognormal diffusions and diffusion-jump processes that are empirically comparable, table 1a implies that the 'jump-component' premium in deep-out-of-the-money options about to expire may not be very significant. The theoretical Merton option prices C_M and P_M confirm this implication.

As expected, the difference between the Black–Scholes prices and their upper bounds increases with the variance and the time to maturity. The former property is obvious. The latter is also plausible, since the impact of 'deviations' of the terminal stock price distribution from lognormality on the option price become more pronounced as more time is allowed to elapse between the current date and the expiration date. With 24 weeks to go, table 1c shows that the upper bound may exceed the Black–Scholes price by as much as \$4.33. Merton jump-diffusion prices behave similarly.

Of course, if enough is known about the two candidate processes to compute their risk-neutral variances, it may be argued that these semi-parametric upper bounds are superfluous, since the option value may be calculated directly under each process. Such upper bounds are, however, still of considerable use for several reasons. Perhaps their main advantage is computational simplicity. They depend only on the mean and variance of the terminal stock price, whereas the various exact option pricing formulas depend intricately on the specific price process. Although evaluating exact option pricing formulas obviously provides sharper price predictions, semi-parametric upper bounds yield one simple measure of the impact of misspecifying the stochastic price process. Indeed, because of the upper bounds' homogeneity and monotonicity, values for them can be easily tabulated for an empirically relevant range of parameters, as in tables 1a–1c. Also, it is almost always computationally simpler to derive the relation of V^* to observable parameters than to actually calculate the option price.[10] Moreover, to calculate the option prices directly, it is first necessary to estimate the parameters anyway. Therefore, computing semiparametric upper bounds yields information at virtually no additional computational cost.

5. Conclusion

I have derived upper bounds for the expected payoff of call and put options that depend only on the mean and variance of the terminal stock price, and not on its entire distribution. Corresponding semi-parametric upper bounds can be derived for option prices when the variance of the associated risk-neutral

[10] The standard Black–Scholes formula is perhaps the only exception. Compare, for example, the computational requirements of deriving (14b) with evaluating the actual option price for the mixed diffusion-jump process (especially when the jump magnitude is not necessarily lognormal).

pricing distribution is known. To implement these bounds empirically, it is necessary to derive the relationship between the risk-neutral pricing distribution and the actual distribution of the terminal stock price. Such a relation may not always be tractable. However, for the two leading classes of price processes used in financial asset-pricing models, lognormal diffusions and mixed diffusion-jump processes, the risk-neutral variance V^* is a simple transformation of the actual variance and can readily be estimated. Because the risk-neutral variances of the lognormal diffusion and the mixed diffusion-jump processes are numerically identical for any given data set, the estimated semi-parametric upper bounds are the same regardless of which class the actual stock price process belongs to. An interesting empirical implication of this result is that, under a plausible range of parameter values, the difference between Black–Scholes option prices and their semi-parametric upper bounds is quite small for deep-out-of-the-money options about to expire, irrespective of the possible presence of a jump component. A comparison of these upper bounds with actual market prices, however, does indicate that large differences exist.[11] This implies that option prices obtained from the standard contingent claims analysis may not be robust to misspecification of the fundamental asset's stochastic law of motion. A direct test of this implication seems to be a promising direction for further investigation.

Appendix: Proof of Proposition 1

The proof of the proposition follows from an ingenious result due to Scarf (1958). For completeness, it is restated here.

Lemma (Scarf). Let c, μ, and δ be fixed. Then there exists a quadratic function $Q(X) = \alpha + \beta X + \gamma X^2$ such that $Q(X) \leq \min[X, c]$ for $X \geq 0$ with equality holding at only two points a and b. Moreover there exists a two-point distribution situated at a and b with mean μ and standard deviation δ.

Proof. See Scarf (1958).

Corollary (Scarf). Let the two-point distribution in the preceding lemma be denoted by $H(X)$. Then $H(X)$ minimizes

$$\int_0^\infty \min[X, c]\, \mathrm{d}P(X), \tag{A.1}$$

over all distributions $P(X)$ with mean μ and standard deviation δ.

[11] See Lo (1986).

Proof.

$$\int_0^\infty \min[X, c]\, \mathrm{d}P(X) = \int_0^\infty (\min[X, c] - Q(X))\, \mathrm{d}P(X)$$

$$+ \int_0^\infty Q(X)\, \mathrm{d}P(X), \tag{A.2}$$

$$\geq \int_0^\infty Q(X)\, \mathrm{d}P(X) = \alpha + \beta\mu + \gamma(\mu^2 + \sigma^2), \tag{A.3}$$

by the above Lemma. But $H(X)$ attains this lower bound since $Q(X)$ and $\min[X, c]$ are equal on those points a and b where H has all its weight. Q.E.D.

From these two results, Scarf demonstrates that

$$\min(\mathrm{E}[\min[X, c]]) = \frac{\mu^2 c}{\mu^2 + \delta^2} \qquad \text{if} \quad c \leq \frac{\mu^2 + \delta^2}{2\mu},$$

$$= \frac{\mu + c}{2} - \tfrac{1}{2}\sqrt{(c - \mu)^2 + \delta^2} \quad \text{if} \quad c > \frac{\mu^2 + \delta^2}{2\mu}, \tag{A.4}$$

where

$$\mu = \mathrm{E}[X] \quad \text{and} \quad \delta^2 = \mathrm{E}[X - \mu]^2,$$

and the minimization is performed over all distributions P with the same mean μ and variance δ^2. Armed with this result, we now prove the proposition:

Proof.

$$\max[X - c, 0] + \min[X, c] = X, \tag{A.5}$$

$$\mathrm{E}(\max[X - c, 0]) = \mu - \mathrm{E}(\min[X, c]), \tag{A.6}$$

$$\max[\mathrm{E}(\max[X - c, 0])] = \mu - \min[\mathrm{E}(\min[X, c])]. \tag{A.7}$$

Thus we have

$$\max[\mathrm{E}(\max[X-c,0])]$$

$$= \frac{\mu^2(\mu-c)+\mu\delta^2}{\mu^2+\delta^2} \qquad \text{if} \quad c \le \frac{\mu^2+\delta^2}{2\mu},$$

$$= \tfrac{1}{2}\left[\mu-c+\sqrt{(c-\mu)^2+\delta^2}\right] \quad \text{if} \quad c > \frac{\mu^2+\delta^2}{2\mu}. \tag{A.8}$$

Eq. (2) follows directly from this. Q.E.D.

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