A new methodology for statistically testing contingent-claims asset-pricing models based on asymptotic statistical theory is proposed. It is introduced in the context of the Black-Scholes option-pricing model, for which some illustrative estimation, inference, and simulation results are also presented. The proposed methodology is then extended to arbitrary contingent claims by first considering the estimation problem for general Itô processes and then deriving the asymptotic distribution of a general contingent claim which depends upon such Itô processes.

1. Introduction

Since Black and Scholes (1973) and Merton (1973, 1976) introduced their now famous option-pricing models, their methodology has been applied to the pricing of a variety of other assets whose payoffs are contingent upon the value of some other underlying or 'fundamental' asset. By assuming that the fundamental asset price process is of the Itô type and that trading takes place continuously in time, the price of a contingent claim can often be derived by using the hedging and no-arbitrage arguments of Black-Scholes and Merton. Since the deduced pricing formulas are almost always functions of unknown parameters of the fundamental asset-price processes, any empirical application of contingent-claims analysis must first consider the statistical estimation of fundamental asset-price parameters. In addition, since parameter estimates are ultimately employed in the pricing formulas in place of the true but unknown parameters, the sampling variation of parameter

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estimates will of course induce sampling variation in the estimated contingent-claims prices about their true values. The practical value of contingent-claims analysis then depends critically on how parameter-estimation errors affect the accuracy of the resulting contingent-claims price estimator. Furthermore, some measure of the induced estimation error is required if the model is to be empirically tested. Indeed, although a number of papers have studied the discrepancies between estimated and observed prices for particular contingent claims, to date there have been few direct statistical tests of these models. In a spirit similar to Gibbons' (1982) examination of the capital-asset-pricing model, this paper proposes a new framework in which tests of contingent-claims asset-pricing models may be performed and in which the accuracy of contingent-claims price estimates may be quantified statistically.

This new approach seems particularly fruitful for several reasons. In particular, it is shown that the suggested methodology can be applied to any contingent claim for which the associated fundamental asset-price parameters may be estimated. Few additional assumptions beyond those common to all contingent-claims models are required in order to apply the proposed methods. In addition, the results derived in this paper are computationally quite simple to implement and, in most applications, require few calculations beyond those performed by standard econometric computer packages. Furthermore, such a framework is well-suited to the standard tools of statistical inference, estimation, and forecasting. In fact, since the distribution of the contingent-claims estimator is derived in closed form, all the usual hypothesis testing and forecasting techniques may be applied to contingent-claims analysis. This is achieved through the use of large-sample or asymptotic statistical theory which essentially consists of applying laws of large numbers and central limit theorems to otherwise intractable estimation and inference problems. By appealing to large-sample arguments, it is possible to derive explicitly the limiting distribution of highly non-linear functions (such as the Black–Scholes

1 Papers by Black and Scholes (1972), Merton (1973), Black (1975), Macbeth and Merville (1979, 1980), and Gultekin, Rogalski and Tinic (1982) have noted systematic differences between observed market prices of call options and prices obtained from the Black–Scholes formula but did not formally test whether such departures were statistically significant. However, several studies have considered testing the efficiency of options markets. In particular, Black and Scholes (1972), Galai (1978), and Finnerty (1978) have explored the possibility of excess returns resulting from observed options prices deviating from the Black–Scholes prices. Chiras and Manaster (1978) study possible excess returns generated by using implied standard deviations in the pricing formula. Whaley (1982) uses implied standard deviations in examining various pricing formulas for calls on stocks with known dividends. Violations of certain boundary conditions by observed market prices have also been investigated by Galai (1978) and Bhattacharya (1983). Although many of these empirical findings are quite striking, without some guidelines as to the statistical significance of observed deviations, hypothesis tests cannot formally be constructed. Even in Whaley's (1982) six regression tests of option valuation, since the linear regression equations are not determined by theoretical considerations there is no guarantee that the subsequent test statistics have a particular sampling distribution. Because Rubinstein's (1985) tests are non-parametric, his results may be the most compelling.
formula) of fundamental parameter estimates. Indeed, if a closed-form solution for the contingent claim's price exists, it is demonstrated that the limiting distribution of price estimators may also be derived in closed form. This not only renders the calculation of standard errors trivial, but also enables comparative static analyses to be performed. An example of this is provided in section 2. In fact, it will be shown that even if a closed-form solution does not exist for the contingent claim's price, the limiting distribution may still be calculated using numerical methods.

For expository purposes, this new methodology is introduced in the context of the Black–Scholes (BS) call-option-pricing model. Of course, since Boyle and Ananthanarayanan (1977) have derived exact small-sample results for this case, the nature of our application is mainly illustrative. Nevertheless, even in this case there are several advantages to the large-sample approach. One significant advantage mentioned above is that since a closed-form expression for the limiting distribution is derived, various comparative static issues, such as how estimation error varies with the stock-price/exercise-price spread, may be resolved analytically. However, the most obvious appeal of asymptotics is its tractability. As an example, consider performing a symmetric two-sided hypothesis test for a particular option. In order to construct such a finite-sample test, critical values for the distribution of the option-price estimator must be determined by numerical integration. In contrast, an asymptotic test relies on the standard normal critical values. Furthermore, in situations where a contingent-claim price function cannot be derived in closed form, numerically determining the finite-sample distribution may be computationally infeasible whereas the corresponding asymptotic distribution may still be derived with little difficulty. Finally, the approach that Boyle and Ananthanarayanan propose does not extend readily to multi-parameter situations nor to cases in which the finite-sample distribution of the fundamental asset-price parameter is unknown. For example, consider estimating the term structure of interest rates in Vasicek's (1977) model for the specific case in which the spot rate follows an Ornstein–Uhlenbeck process \[eq. (19)\]. Vasicek derives a closed-form expression for bond prices as a function of the parameters \( \theta = (\alpha, \gamma, \beta) \), hence estimates for bond prices may be computed by inserting the estimators \( \hat{\theta} = (\hat{\alpha}, \hat{\gamma}, \hat{\beta}) \) in the pricing formula. To deduce the finite-sample distribution of bond prices then, the finite-sample distribution of the estimators is required. To this author's knowledge, there do not exist estimators \( \hat{\theta} \) in this case for which the finite-sample distribution is known. However, these parameters are easily estimated via maximum likelihood and therefore have well-defined asymptotic distributions. The limiting distribution for bond price estimates is then readily determined as outlined in section 4.

In order to illustrate this new methodology and also to clarify the particular econometric issues at hand, section 2 derives the large-sample properties of the BS call-option-price estimator. The derived asymptotic statistics are then
calculated using data for options written on a particular stock, and some simple hypothesis tests are performed. To explore the accuracy of the proposed estimators, some simulation evidence is presented in section 3. In section 4 the methodology is developed in its most general form, and we conclude in section 5.

2. Estimation and inference for the BS call-option-pricing model

Let $S(t)$ denote the price of a stock at time $t$ and let $F(S, E, r, \tau, \sigma^2)$ be the price of a corresponding call option at time $t$ with exercise price $E$ and time-to-maturity $\tau$, where $r$ is the interest rate on a default-free pure discount bond with time-to-maturity $\tau$ and $\sigma^2$ is the variance rate of the underlying stock-price process $S(t)$. Under the assumptions of the BS model, $F$ is determined by the well-known formula

$$F = S\Phi(d_1) - Ee^{-r\tau}\Phi(d_2), \quad (1a)$$

$$d_1 = \frac{1}{\sigma\sqrt{\tau}} \left[ \ln\left( \frac{S}{E} \right) + (r + \frac{1}{2}\sigma^2)\tau \right], \quad (1b)$$

$$d_2 = d_1 - \sigma\sqrt{\tau}, \quad (1c)$$

where $\Phi$ is the standard normal cumulative distribution function. Although the stock price, exercise price, time-to-maturity, and interest rate are in principle observable without error, the variance $\sigma^2$ of the underlying stock is unknown. The following analysis takes as its starting point the assumption that the stock-price process $S(t)$ is the usual lognormal diffusion process given by

$$dS/S = \mu\,dt + \sigma\,dW. \quad (2)$$

The BS model is not assumed to obtain, but instead forms the null hypothesis which is to be tested. Since an estimate $\hat{\sigma}^2$ of $\sigma^2$ may be obtained by using historical data, evaluating $F$ at $\hat{\sigma}^2$ yields an estimate of the corresponding option price. Although the resulting option estimator is clearly not unbiased, it is consistent if the variance estimator is consistent. Consistency is a particularly desirable property since by definition a consistent estimator approaches the true value with probability one as the sample size grows. This is distinct from an unbiased estimator which, although correct on average, may fluctuate considerably about its true value even in very large samples.2

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2As an extreme example, consider a coin which has an unknown probability $p$ of coming up 'Heads' when tossed, where $p$ is known to be between 1/4 and 3/4. Toss the coin once and consider the estimator $\hat{p}$ which equals 1 if the coin comes up 'Heads' and 0 if it comes up 'Tails'. Although this estimator is incorrect with probability one, it is in fact unbiased. This rather contrived example illustrates the inadequacy of using unbiasedness as the sole criterion for choosing an estimator: its variance must also be considered.
Given a consistent estimator of the option price, a direct statistical test of the BS model can be constructed by comparing this estimate with the actual market price. Since the estimated price is subject to sampling variation, a measure of its 'spread' is needed in order to perform a meaningful comparison. More formally, a test of whether or not the estimated option price differs significantly from the actual market price requires the calculation of the standard error about the estimated option price and the estimator's sampling distribution. In this section, the asymptotic distribution of the option-price estimator is derived and is used to compare actual market price with their BS estimates.

2.1. Estimation and asymptotic distribution of call-option prices

Estimation of the stock-price dynamics is considered first. Suppose that \( n + 1 \) equally spaced observations of \( S(t) \) are taken in the time interval \([0, T]\). Letting \( h = T/n \), Rosenfeld (1980) has shown that the maximum likelihood (ML) estimator of \( \sigma^2 \) is given by

\[
\hat{\sigma}_{ML}^2 = \frac{1}{T} \sum_{k=1}^{n} \left( X_k - \frac{1}{n} \sum_{j=1}^{n} X_j \right)^2,
\]

where \( X_k \) is the log of the price-relative \( S(kh)/S((k - 1)h) \). Under mild regularity conditions, it is well-known that the general ML estimator is consistent, asymptotically normally distributed, and efficient in the class of all consistent and uniformly asymptotically normal (CUAN) estimators.\(^3\) In addition, the ML estimator of any well-behaved non-linear function of a given parameter is simply the non-linear function of the ML estimator of that parameter. That is, the ML estimator \( \hat{F}_{ML} \) of the option price \( F \) may be obtained by evaluating \( F \) at \( \hat{\theta}_{ML} \). Since \( \hat{F}_{ML} \) is a true ML estimator, it also exhibits the usual maximum-likelihood properties cited above.

Since the option estimator \( \hat{F}_{ML} \) depends on the estimator \( \hat{\sigma}_{ML}^2 \), the asymptotic distribution of \( \hat{F}_{ML} \) is related to the asymptotic distribution of \( \hat{\sigma}_{ML}^2 \). It may easily be shown that \( \hat{\sigma}_{ML}^2 \) has the following asymptotic distribution: \(^4\)

\[
\sqrt{n} \left( \hat{\sigma}_{ML}^2 - \sigma^2 \right) \overset{\Delta}{\sim} N(0, 2\sigma^4).
\]

where the notation \( \overset{\Delta}{\sim} \) indicates that this is an asymptotic relation. Now consider the estimator \( \hat{F}_{ML} \) as a function of \( \hat{\sigma}_{ML}^2 \), holding all other arguments

\(^3\)For perhaps the weakest set of regularity conditions which insure consistency and asymptotic efficiency of maximum-likelihood estimators, see Huber (1967).

\(^4\)See Kendall and Stuart (1979).
fixed, i.e., \( \hat{\sigma}_{ML}^2 = F(\sigma_{ML}^2) \). The asymptotic distribution of \( \hat{\sigma}_{ML}^2 \) may then be derived by applying standard statistical limit theorems to the Taylor series expansion of \( F(\hat{\sigma}_{ML}^2) \) about the true parameter \( \sigma^2 \), yielding the desired result [see Rao (1973)]:

\[
\sqrt{n} \left( \hat{\sigma}_{ML}^2 - \sigma^2 \right) \sim N \left( 0, 2\sigma^4 \left( \frac{\partial F(\sigma^2)}{\partial \sigma^2} \right)^2 \right).
\]

That is, for a sufficiently large number \( n \) of observations,\(^5\) the sampling distribution of \( \hat{\sigma}_{ML}^2 \) is approximately normal with mean \( \sigma^2 \) and variance \( (2\sigma^4/n)(\partial F(\sigma^2)/\partial \sigma^2)^2 \equiv V_F \). Given the BS pricing formula (1), the quantity \( V_F \) may be calculated explicitly as

\[
V_F = \frac{1}{2n} S^2 \sigma^2 \phi^2(d_1),
\]

where \( \phi \) is the standard normal density function.

2.2. Historical versus implied variance estimators

In contrast to estimating variances with historical data, several studies have indicated that variances implicit in option prices seem to be better estimators in several ways [see for example Latané and Rendleman (1976)]. Loosely speaking, this may be because historical estimates are ‘retrospective’ whereas implicit estimates are ‘prospective’. That is, since option prices are determined daily, all current information affecting (among other things) the volatility of the underlying stock price will be impounded in those prices. For example, new information which changes the current expectation of future volatilities will, in an efficient market, be reflected in observed option prices but will obviously not be evident from historical estimates.

\(^5\)It is assumed that \( h \) is constant as \( n \) increases so that \( T \) also increases. If, instead, \( T \) is kept constant while \( n \) increases and \( h \) decreases, one of the regularity conditions will be violated. In this case, the estimator need not approach the true parameter as \( n \) increases. For example, in estimating the parameters of a lognormal diffusion process, Merton (1980) and Rosenfeld (1980) observe that the accuracy of the drift-rate estimator (as measured by its variance) does not increase with more frequent observations if \( T \) is fixed. This seeming contradiction to the asserted consistency of the ML estimators is resolved by observing that when \( T \) is fixed and \( n \) increases without bound, the regularity condition which requires that the information matrix approach a non-singular matrix [see Rao (1973)] is violated. This, however, does not imply that ML is not applicable but rather that its asymptotic properties have been misinterpreted. Specifically, the asymptotic results used in proving the consistency of ML require that \( n \) increases without bound *ceteris paribus*, i.e., holding other parameters (\( \mu, \sigma^2, h \)) constant. Note that if these parameters were in fact held constant, increasing \( n \) will indeed increase the accuracy of the drift estimator. This example thus illustrates the importance of checking the regularity conditions when applying asymptotic results to non-standard situations, such as taking more frequent observations in a fixed time interval.
Implicit in those studies are two critical assumptions. First, a specific option-pricing model must be known to obtain. Second, the options markets must be known to be efficient. Under these two assumptions, implicit estimators are clearly preferred. However, the approach taken in this paper is fundamentally different. The only assumption required in order to apply the general methodology proposed in this paper involves the stochastic specification of the underlying asset price. In particular, it is assumed (in the general case of section 4) that the price process may be described by a first-order non-linear stochastic differential equation driven by both standard white noise and Poisson components, i.e., general Itô processes. This, of course, is a special case of more general processes described by higher-order stochastic differential equations, which again may be considered special cases of still more general processes. But because empirically asset prices seem to be well represented by the class of Itô processes, assuming this particular form of dynamics may be justified to some extent. This paper then suggests a method of testing models based on this assumption. However, in computing implicit variances, a specific model has already been assumed to obtain. Therefore, using implicit variances in this study is inappropriate. More specifically, a test of the BS model based on implied variances which presupposes that the BS model obtains will always confirm the model.

2.3. Comparative static analysis of the asymptotic variance $V_F$

The expression for $V_F$ in eq. (6) is of interest for several reasons. In addition to providing a measure of option-price estimators' dispersion in large samples, the analytic formula for $V_F$ may also be used to examine how changes in the underlying parameters affect the option estimates. As an example, consider the systematic biases of the BS prices noted in several empirical studies. Macbeth and Merville (1979) observe that in-the-money call options are underpriced by the BS formula and vice-versa for out-of-the-money calls, and that the degree of mispricing is aggravated by the spread between stock and exercise price for most options. Black (1975), Merton (1976), and Gultekin, Rogalski and Tinic (1982) observe essentially the opposite biases. To see whether such biases may be explained merely by sampling variation, consider the derivatives of $V_F$ with respect to the stock and exercise prices and the time-to-maturity:

\[
\frac{\partial V_F}{\partial S} = -\frac{1}{n} S \sigma \sqrt{\tau} \phi^2(d_1) d_2
\]  
\[
\frac{\partial V_F}{\partial E} = \frac{1}{n} S^2 \frac{\sigma \sqrt{\tau} \phi^2(d_1) d_1}{E}
\]  
\[
\frac{\partial V_F}{\partial \tau} = \frac{1}{2n} S^2 \phi^2(d_1) \sigma^2 \left[ 1 - \frac{1}{\sigma^2 \tau} \left[ \tau + \frac{1}{2} \sigma^2 \right]^2 + \frac{1}{\sigma^2 \tau} \left[ \ln \frac{S}{E} \right]^2 \right].
\]
Let
\[ k_1 = e^{-(r+\frac{1}{2}\sigma^2)\tau}, \quad k_2 = e^{-(r+\frac{1}{2}\sigma^2)\tau}, \quad k_3 = e^{(r+\frac{1}{2}\sigma^2)\tau} = 1/k_2. \]

The following inequalities are then established:
\begin{align*}
\frac{\partial V_F}{\partial S} \geq 0 & \quad \text{iff} \quad \frac{S}{E} \leq k_1, \quad (8a) \\
\frac{\partial V_F}{\partial E} \geq 0 & \quad \text{iff} \quad \frac{S}{E} \geq k_2. \quad (8b)
\end{align*}

Although obtaining a similar pair of equivalent inequalities for \( \frac{\partial V_F}{\partial \tau} \) does not seem possible, two useful sufficient conditions for the monotonicity of the derivative can be derived:\(^6\)
\begin{align*}
\frac{\partial V_F}{\partial \tau} > 0 & \quad \text{if} \quad \frac{S}{E} < k_2 \quad \text{or} \quad \frac{S}{E} > k_3, \quad (8c) \\
& \quad \text{or} \quad \frac{S}{E} \geq k_4 = \frac{1}{\sqrt{\tau}} (r + \frac{1}{2}\sigma^2) < 1. \quad (8d)
\end{align*}

Note that condition (8d) does not depend upon the stock or exercise prices. Even if (8d) is not satisfied, monotonicity still obtains if the option is sufficiently in or out of the money which is the essence of condition (8c). In table 1, values of \( k_1, k_2, k_3, \) and \( k_4 \) have been tabulated for various times-to-maturity measured in weeks, given an (annual) interest rate of 10 percent and an (annual) standard deviation of 50 percent. Several observations may be made from the values in table 1. Since the interval \([k_2, k_3]\) is fairly concentrated about 1.0, an increase in the time-to-maturity will increase the variance about the option-price estimate unless the option is very nearly at-the-money. For example, if the stock price is \$40, then options which are either in- or out-of-the-money by \$5 or more are more precisely estimated as the time-to-maturity declines. This may well explain Macbeth and Merville's (1979, 1980) finding that biases of in- and out-of-the-money options decrease as the time to expiration decreases. Furthermore, the miniscule values for \( k_4 \) in table 1 indicate that the variance of option-price estimators will increase with the time-to-maturity even for at-the-money options. This would also support Gultekin, Rogalski and Tinic's (1982) observation that 'in general, the [BS] formula gives much less accurate estimates for long-lived options'.

\(^6\) I am grateful to Jay Shanken for pointing out condition (8d).
Table 1

Values of \( k_1, k_2, k_3, \) and \( k_4 \) for various time-to-maturity \( \tau \) (measured in weeks), where an annual interest rate of 10% and an annual standard deviation of 50% are assumed, corresponding to weekly values of \( r = 0.00183 \) and \( \sigma = 0.06934 \).

\[
\begin{align*}
\tau & \quad k_1 = \exp - (r - \frac{1}{2} \sigma^2) \tau & \quad k_2 = \exp - (r + \frac{1}{2} \sigma^2) \tau & \quad k_3 = \exp(r + \frac{1}{2} \sigma^2) \tau & \quad k_4 = \frac{\sqrt{r}}{\sigma} (r + \frac{1}{2} \sigma^2) \\
1.00 & \quad 1.00057 & \quad 0.99577 & \quad 1.00425 & \quad 0.00847 \\
5.00 & \quad 1.00286 & \quad 0.97904 & \quad 1.02141 & \quad 0.01895 \\
9.00 & \quad 1.00515 & \quad 0.96259 & \quad 1.03887 & \quad 0.02542 \\
13.00 & \quad 1.00745 & \quad 0.94641 & \quad 1.05662 & \quad 0.03055 \\
26.00 & \quad 1.01496 & \quad 0.89570 & \quad 1.11645 & \quad 0.04320 \\
\end{align*}
\]

Table 2

Estimated values of \( k_1, k_2, k_3, \) and \( k_4 \) for Litton stock. The (annual) interest rate used is 9.443\%, the average return on 26-week Treasury bills quoted in the January 15, 1979 issue of the Wall Street Journal, resulting in the weekly value \( r = 0.00174 \), and the estimated weekly variance \( \sigma^2 \) is 0.00423.

\[
\begin{align*}
\tau^w & \quad \tilde{k}_1 = \exp - (r - \frac{1}{2} \tilde{\sigma}^2) \tau & \quad \tilde{k}_2 = \exp - (r + \frac{1}{2} \tilde{\sigma}^2) \tau & \quad \tilde{k}_3 = \exp(r + \frac{1}{2} \tilde{\sigma}^2) \tau & \quad \tilde{k}_4 = \frac{\sqrt{r}}{\tilde{\sigma}} (r + \frac{1}{2} \tilde{\sigma}^2) \\
1.00 & \quad 1.00038 & \quad 0.99615 & \quad 1.00386 & \quad 0.00821 \\
5.00 & \quad 1.00190 & \quad 0.980931 & \quad 1.01944 & \quad 0.01836 \\
9.00 & \quad 1.00343 & \quad 0.965938 & \quad 1.03526 & \quad 0.02463 \\
13.00 & \quad 1.00495 & \quad 0.951174 & \quad 1.05133 & \quad 0.02960 \\
26.00 & \quad 1.00993 & \quad 0.904732 & \quad 1.08530 & \quad 0.04186 \\
\end{align*}
\]

\( ^{a} \)Times-to-maturity \( \tau \) are measured in weeks.
Another property of the option-price estimator implied by the values in table 1 is that, loosely speaking, if an option is deep in-the-money \((S/E > 1)\), then as the exercise price increases, so will the variance about the option-price estimate. If an option is deep out-of-the-money \((S/E < 1)\), then decreasing the exercise price increases the variance of the estimated option price. In other words, option-price estimates exhibit more variation for either deep in or out of the money options as the exercise price shifts closer to the prevailing stock price. Of course, these statements may be made precise by computing the specific values of \(k_1\) and \(k_2\), for particular options of interest.

\[ \text{2.4. Statistical tests of the BS option-pricing model} \]

The most direct application of the quantity \(V_F\) is in statistically testing the BS model. In particular, consider the null hypothesis that the BS model obtains. Letting \(\bar{F}\) denote the observed market option price, this null hypothesis may be stated as

\[ H_0: \quad F(\sigma^2) = \bar{F}. \]  

(9)

This hypothesis may then be tested by computing the statistic

\[ q = \frac{[F(\hat{\sigma}^2_{ML}) - \bar{F}]/\sqrt{\hat{V}_F}}. \]  

(10)

Since \(V_F\) depends upon the unknown parameter \(\sigma^2\), a corresponding ‘\(t\)-statistic’ \(z\) may be calculated by using a consistent estimator \(\hat{V}_F = V_F(\hat{\sigma}^2_{ML})\) in place of \(V_F\) in eq. (10). Note that the resulting statistic is still asymptotically standard normal. The test is then performed by rejecting \(H_0\) if \(z\) lies outside an acceptable range of 0 and accepting otherwise, where the range of acceptability is determined by the desired size of the test. For example, if \(z\) falls outside the interval \([-1.96, 1.96]\) then \(H_0\) may be rejected at the 5% level. In addition, the usual forms of conditional forecasting and confidence-interval calculations may be performed given the estimated variance.

\[ \text{2.5. An empirical example} \]

Because the expression for \(V_F\) is analytically quite simple, computing standard errors for option-price estimates requires little calculation beyond the estimation of the stock-price volatility. As an illustrative example, standard errors and the associated \(z\) statistics have been computed in table 3 for traded options on Litton stock for January 12, 1979. Litton was chosen arbitrarily.

\[ ^7 \text{Of course, the distribution of } z \text{ is not the Student's } t \text{ since the numerator and denominator are not statistically independent. However, asymptotically it is normally distributed.} \]
Table 3

Maximum-likelihood estimates of prices ($\hat{P}$) and asymptotic standard deviations ($\sqrt{\hat{V}_P}$) of Litton call options traded on January 12, 1979, and estimates of the corresponding $z$-statistics [$z = (\hat{P} - \hat{F})/\sqrt{\hat{V}_P}$ where $\hat{F}$ is the observed market price of the option]. The maximum-likelihood estimate for the variance of Litton stock is taken from Rosenfeld's (1980) study which used 312 weekly observations from January 1973 to December 1978, and is given by $\hat{\sigma}^2 = 0.00254$. The stock prices $S$, exercise prices $E$, and option prices $\hat{F}$ were obtained directly from the January 15, 1979 issue of the Wall Street Journal. The interest rate used was the average return on 26-week Treasury bills for which the same issue of the Wall Street Journal reported an annual rate of 9.443%.

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<th>$S$</th>
<th>$E$</th>
<th>$\tau^*$</th>
<th>$\hat{P}$</th>
<th>$\hat{F}$</th>
<th>$\sqrt{\hat{V}_P} \times 100$</th>
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<td>21.500</td>
<td>30.000</td>
<td>22.00</td>
<td>0.688</td>
<td>0.669</td>
<td>7.5195</td>
<td>-0.25</td>
</tr>
<tr>
<td>21.500</td>
<td>24.375</td>
<td>29.00</td>
<td>-d</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>21.500</td>
<td>15.000</td>
<td>29.00</td>
<td>-c</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
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<td>20.000</td>
<td>29.00</td>
<td>4.625</td>
<td>4.226</td>
<td>10.4769</td>
<td>-3.81</td>
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<td>25.000</td>
<td>29.00</td>
<td>2.375</td>
<td>2.138</td>
<td>11.9523</td>
<td>-1.98</td>
</tr>
<tr>
<td>21.500</td>
<td>30.000</td>
<td>29.00</td>
<td>-d</td>
<td>-</td>
<td>-</td>
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</tr>
</tbody>
</table>

a Time-to-maturity is measured in weeks.
b Under the joint null hypothesis that the Black–Scholes options-pricing model obtains and that the stock price follows a lognormal diffusion, the $z$-statistic is asymptotically standard normal. Therefore, the null hypothesis may be rejected at the 5% level if the estimated $z$-statistic falls outside the interval $[-1.96, 1.96]$.
c Not traded.
d No option offered.
from a subset of five non-dividend paying stocks for which Rosenfeld (1980) estimated drift and variance coefficients according to the dynamics given by eq. (2). The estimate of the stock's variance was obtained from Rosenfeld (1980) and was estimated using 312 weekly observations from the period January 1973 to December 1978. The interest rate used was the (annual) 26-week Treasury-bill rate quoted on January 15, 1979 in the \textit{Wall Street Journal} (9.443%). In table 2, the estimated values of $k_1$, $k_2$, $k_3$, and $k_4$ for various times-to-maturity are presented for Litton. The values of $k_4$ indicate that (8d) is satisfied, hence monotonicity of the variance with respect to the time-to-maturity holds for all options of Litton. Note that, holding the exercise price constant, the standard error of every estimated option price in table 3 increases with an increased time-to-maturity. Also, whether or not an option is in- or out-of-the-money does not seem to be systematically related to whether it is underpriced or not. Of course, previous empirical studies have used a much larger set of options than the few considered here, so the lack of discernible patterns in table 3 is not conclusive.

The z-statistics seem to indicate that the data are inconsistent with the null hypothesis $H_0$ that the BS model obtains. For example, out of the eleven options written on the Litton stock, only two estimates had standard errors outside the 1%-critical region and only one estimate had its standard error outside the 5%-critical region. However, caution must be exercised in interpreting this since for each stock the tests are certainly not independent. Nevertheless, a simultaneous test of $H_0$ for all Litton options with a nine-week time-to-maturity results in rejection at the 5% level of significance.\footnote{I am grateful to Eric Rosenfeld for providing me with the variance estimates for these stocks.} Of course, the empirical application presented in table 3 is meant only as an illustration of the general methodology and not as a conclusive test of the BS model. However, the results do seem to agree with more comprehensive tests of the BS model such as Rubinstein's (1985) exhaustive study.

It is important to note that the above test of $H_0$ is in fact a \textit{joint} test of the BS option-pricing model and of the associated stock-price dynamics. Rejecting $H_0$ in this case may not necessarily imply that the BS model does not obtain. However, because the BS formula is so closely related to the particular form of the stock-price dynamics it is difficult to imagine a situation in which (1) obtains but (2) does not. In fact, Rosenfeld (1980) has tested the hypothesis that these three stocks follow the process (2) and rejects in favor of a combined lognormal diffusion and jump process. But in this situation, the model outlined in (1) does not obtain and must be modified along the lines

\footnote{Specifically, using the Bonferroni correction for the simultaneous testing of five hypotheses at the 5 percent level, the appropriate critical values for a two-sided test is $\pm 2.58$ (corresponding to a tail probability of slightly less than 2.5/5 percent). For all five options, the associated z-statistic falls within the critical region hence the simultaneous hypothesis may be rejected at the 5 percent level.}
Merton (1976) develops. In addition to the possibility of jumps, Rosenfeld (1980) and Marsh and Rosenfeld (1983) consider several other alternatives which may support the results in table 3. In particular, one plausible explanation is that the variance parameter is not constant over the sample period (6 years). The empirical non-stationarity of stock-price volatilities is a well-known phenomenon which may very well lead to a rejection of the lognormal diffusion (2) as the correct stochastic specification. Of course, in this case the BS model given in (1) is also inappropriate. However, such non-stationarities do not vitiate the usefulness of asymptotic methods for statistical inference but simply suggest that there may exist a better stochastic specification for the asset-price dynamics such as Cox's (1975) constant elasticity of variance (CEV) process. In fact, the stationarity assumption may be statistically tested by applying large-sample results (such as a likelihood-ratio test). Of course, the CEV specification implies a different option-pricing formula than (1) and, although it is not pursued in this paper, a test of that model may also be constructed readily in the framework proposed here.

3. Simulation evidence

Although the empirical evidence presented in section 2 is of interest in its own right, it also illustrates the practical relevance of asymptotic statistical theory to the estimation of general contingent-claims prices. Section 4 demonstrates formally that this methodology may in fact be applied to any other contingent claim provided that its corresponding underlying fundamental asset-price process may be estimated. However, an important issue which determines the usefulness of large-sample results is the number of observations required for those results to obtain. Unfortunately, no general guidelines exist, so this issue must be resolved for each application individually. Nevertheless the increasing sophistication of statistical software coupled with the rapid decline of computer costs allow researchers to determine what constitutes a large sample for a particular estimator relatively easily.

3.1. Design of experiments

In this section, a simple simulation study is conducted for the call-option-price estimators proposed in section 2. Each Monte Carlo experiment involves generating a time series for the stock-price process with a given drift and variance rate using a random number generator, and then computing price estimates and corresponding asymptotic standard-deviation estimates for hypothetical options written on that stock. The estimated option price and

10 The random number generator used was the subroutine GGNQF in the IMSL software package. All computations were done in double precision FORTRAN on a Digital VAX 11/780.
asymptotic standard deviation may then be compared with their true values. This procedure is repeated 1000 times in order to deduce the finite-sample properties of the estimators. By varying the length of the stock-price series generated for the 1000 replications and noting its effect upon the estimators' sampling behavior, it is possible to deduce the minimum number of observations required to insure that the associated asymptotic statistics are adequate approximations. By varying other parameters, it is also possible to study how the asymptotic approximation to finite-sample properties may be related to the terms of an option contract such as the time-to-maturity or the stock-price/exercise-price spread. Throughout the simulations, the following parameter values were assumed and held constant:

\[ S = 40, \]
\[ \sigma^2 = 0.5200 \text{ annual}, \]
\[ r = 0.1000 \text{ annual}. \]

The simulations were carried out at the weekly frequency for which \( r \) and \( \sigma^2 \) were adjusted appropriately.

3.2. Simulation results

Tables 4 and 5 summarize the finite-sampling properties of the option-price and asymptotic-variance estimators across the 1000 replications for various options and stock-price sample sizes. Each table corresponds to experiments with hypothetical options of the same time-to-maturity \( T \). Tables 4a and 4b report simulation results for hypothetical options which are at-the-money and in- and out-of-the-money by $5 for maturities 1 and 13 respectively. Tables 5a and 5b display simulation results for options which are in- and out-of-the-money by $15 with, respectively, 1 and 13 weeks to go. Experiments with options of intermediate exercise prices, times-to-maturity other than 1 and 13, and assorted stock-price/interest-rate/stock-price-variance combinations were also conducted but since the results depicted in tables 4 and 5 are generally confirmed in these other experiments, in the interest of brevity those results are not reported here.\(^{11}\)

Within each table, every row corresponds to a separate and independent experiment. That is, each experiment is based completely on newly generated data and uses no data generated in other experiments. Each experiment

\(^{11}\)The actual proportions of the \( z \)-statistics in the 5 and 10 percent tail regions (i.e., the actual size of 5 and 10 percent tests) were also tabulated and generally agreed with the results in tables 4 and 5. That is, in those experiments for which the studentized range did not reject normality, the actual tail regions were not statistically different from the theoretical 5 and 10 percent values. The complete set of simulation results are available from the author upon request.
involves simulating a time series of stock prices of a given length (sample size), computing the estimators $\hat{F}_{ML}$ and $\hat{V}_F$, and the test statistic $z$ for a particular hypothetical option, repeating this 1000 times, tabulating the subsequent sampling distribution for the estimators and $z$, and finally testing the standard normality of $z$. Although the estimators $\hat{F}_{ML}$ and $\hat{V}_F$ may also be checked for normality, for purposes of hypothesis testing and constructing confidence intervals the standard normality of the statistic $z$ is more relevant. Of the many tests for departures from normality, only two are considered here. The first is the usual $\chi^2$-test of goodness-of-fit which measures the 'distance' between the hypothesized distribution function (normal) and the empirical distribution function. The second is the studentized range test which is more sensitive to departures from normality in the tails of the distribution. Since the primary use of $z$ is in the testing of hypotheses, departures from normality in the tail areas are of more concern than differences in the center of the distribution. For this reason, the results of the studentized range test may be of more consequence than the $\chi^2$-test. Both tests are performed and the results are given in the last two columns of each row.

Consider the entries in table 4a. The first five rows comprise the simulation evidence for a call option with exercise price $35 and one week to maturity. The second five rows correspond to the experiment of a call option with exercise price $40 also maturing in one week, and the last five rows are results for a call with exercise price $45 and one week to go. The first column indicates the length of the stock-price series generated by the random number generator. The second, third, and fourth columns display, respectively, the true or population value of the option, the mean of the option estimator across the 1000 replications, and the bias in percentage terms. The standard deviation of the option estimate across the replications is given in parentheses under the option estimate. The fifth, sixth, and seventh columns present the simulated population-variance value, the mean of the asymptotic variance estimator ($\hat{V}_F$), and percentage bias, respectively. The eighth column provides the mean and standard deviations of the $z$-statistic over all the replications. In the last three columns, statistics which indicate how close $z$ is to a standard normal variate are displayed. The first is the $\chi^2$-test with the $p$-value given in parentheses below the test-statistic. The next column displays the skewness coefficient of $z$ across the replications and the last column presents the studentized range of $z$.

As Boyle and Ananthanarayanan (1977) have shown, for an at-the-money call option few observations are required in order to trivialize the bias of the option price estimator. The largest absolute price bias observed in tables 4a and 4b, where options are either at-the-money or in- or out-of-the-money by $5, is 0.64%. In addition, in tables 4a and 4b the bias in estimating the variance is also small, the largest being 8.13%. Note that, although on average the bias for both estimators decreases as the length of the stock-price series
Table 4a

Means ($\tilde{F}$, $\tilde{V}_p$), standard deviations, and percentage biases for the sampling distribution of the call-option-price estimator ($\hat{F}$) and its asymptotic variance estimator($\hat{V}_p$) and normality tests for the sampling distribution of the standardized call-option-price estimators ($z$) for options with exercise prices $E = 35, 40, 45$ for stock price $S = 40$, and fixed time-to-maturity $\tau = 1$ (week). For each option, three experiments were performed corresponding to sample sizes of 100, 300, 500, respectively. All experiments involve 1000 replications and assume an (annual) interest rate of 10% (yielding a continuously-compounded weekly rate of 0.183%) and a (weekly) variance rate of 1%.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$F$ (theoretical value)</th>
<th>$\hat{F}$ (std. dev.)</th>
<th>% bias ($\hat{F}$)</th>
<th>$V_p$ (population value)</th>
<th>$\tilde{V}_p$ (std. dev.)</th>
<th>% bias ($\tilde{V}_p$)</th>
<th>$z$ (std. dev.)</th>
<th>$\chi^2$-test* (p-value)</th>
<th>Skewness**</th>
<th>Studentized range***</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E = 35$</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>5.2156 (4.10 $\times 10^{-2}$)</td>
<td>5.2145 (6.98 $\times 10^{-4}$)</td>
<td>0.02</td>
<td>1.7915 $\times 10^{-3}$</td>
<td>-6.76</td>
<td>-0.2387 (1.09)</td>
<td>265.5</td>
<td>-0.803 (0.00)</td>
<td>6.61</td>
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<tr>
<td>300</td>
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<td>5.2151 (1.33 $\times 10^{-4}$)</td>
<td>0.01</td>
<td>5.9285 $\times 10^{-4}$</td>
<td>-6.46</td>
<td>-0.1348 (1.01)</td>
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<td>-0.490 (0.00)</td>
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<tr>
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<td>3.6195 $\times 10^{-4}$</td>
<td>1.33</td>
<td>-0.0087 (1.03)</td>
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<td>-0.410 (0.09)</td>
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<tr>
<td>100</td>
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<td>1.6297 (1.82 $\times 10^{-3}$)</td>
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<td>1.2724 $\times 10^{-2}$</td>
<td>2.06</td>
<td>-0.0811 (1.03)</td>
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<td>-0.422 (0.01)</td>
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<td>-2.08</td>
<td>-0.0226 (0.99)</td>
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<td>-0.0812 (0.99)</td>
<td>50.9</td>
<td>-0.061 (0.98)</td>
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<tr>
<td>$E = 45$</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>100</td>
<td>0.2580 (6.04 $\times 10^{-2}$)</td>
<td>0.2569 (1.23 $\times 10^{-3}$)</td>
<td>0.44</td>
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<td>-0.317 (0.02)</td>
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*The percentage bias is computed as 100 x (true $F$ - $\hat{F}$)/true $F$.

**The percentage bias is computed as 100 x (population $V_p$ - $\tilde{V}_p$)/population $V_p$). Note that the "population variance" is simply the variance of the simulated population and is the square of the standard deviation of $F$ reported in the third column. The percentage bias of $\tilde{V}_p$ was also computed with respect to the theoretical asymptotic variance and in almost every case was smaller than the biases reported in this column. For purposes of inference, the bias with respect to the simulated population variance is more relevant.

***The $\chi^2$-test was performed using 50 equiprobable categories yielding 49 degrees of freedom. A p-value of 0.00 indicates that the actual p-value is less than 0.005.

For 1000 replications, the tabulated 90% and 98% confidence intervals (equal weights in each tail) for the sample skewness statistic are [1.072, 0.127] and [-0.180, 0.180], respectively [see Pearson and Hartley (1972)].

For 1000 replications, the tabulated 90% and 95% confidence intervals (equal weights in each tail) for the sample studentized range statistic are [3.39, 7.35] and [3.68, 7.34], respectively [see Fama (1976)].
Means ($\bar{F}$, $\bar{V}$), standard deviations, and percentage biases for the sampling distribution of the call-option-price estimator ($F$) and its asymptotic variance estimator ($V$) and normality tests for the sampling distribution of the standardized call-option-price estimators ($z$) for options with exercise prices $E = 35, 40, 45$ for stock price $S = 40$, and fixed time-to-maturity $r = 13$ (weeks). For each option, three experiments were performed corresponding to sample sizes of 100, 300, 500, respectively. All experiments involve 1000 replications and assume an (annual) interest rate of 10% (yielding a continuously-compounded weekly rate of 0.183%) and a (weekly) variance rate of 1%.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$F$ (theoretical value)</th>
<th>$\bar{F}$ (std. dev.)</th>
<th>$%$ bias ($\bar{F}$)</th>
<th>$\bar{V}$ (population value)</th>
<th>$%$ bias ($\bar{V}$)</th>
<th>$z$ (std. dev.)</th>
<th>$\chi^2$-test</th>
<th>Skewness</th>
<th>Studentized range</th>
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<td>$E = 35$</td>
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<tr>
<td>100</td>
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<td>8.6958 (0.337)</td>
<td>0.15</td>
<td>1.1356 x 10^1</td>
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<td>0.0191</td>
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<tr>
<td>100</td>
<td>6.1384</td>
<td>6.1146 (0.393)</td>
<td>0.39</td>
<td>1.5460 x 10^1</td>
<td>1.5516 x 10^1</td>
<td>-0.36</td>
<td>0.1303</td>
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<td>5.3426 x 10^2</td>
<td>5.1898 x 10^2</td>
<td>2.86</td>
<td>0.0064</td>
<td>60.7</td>
<td>0.215</td>
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<td>6.1352 (0.172)</td>
<td>0.05</td>
<td>2.9534 x 10^2</td>
<td>3.1146 x 10^2</td>
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<td>0.0472</td>
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<td>-0.243</td>
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<tr>
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<td>4.2075 (0.405)</td>
<td>0.64</td>
<td>1.6409 x 10^2</td>
<td>1.6356 x 10^2</td>
<td>0.32</td>
<td>0.1426</td>
<td>83.0</td>
<td>0.402</td>
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<tr>
<td>300</td>
<td>4.2346</td>
<td>4.2275 (0.229)</td>
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<td>5.2633 x 10^2</td>
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<td>0.0712</td>
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<td>-0.015</td>
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<tr>
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<td>4.2230 (0.184)</td>
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<td>3.2780 x 10^2</td>
<td>3.2783 x 10^2</td>
<td>-0.07</td>
<td>0.0704</td>
<td>64.8</td>
<td>-0.283</td>
</tr>
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</table>

* The percentage bias is computed as 100 x (true $\bar{F}$ - $\bar{F}$)/true $\bar{F}$.
* The percentage bias is computed as 100 x (population $\bar{V}$ - $\bar{V}$)/population $\bar{V}$. Note that the population variance is simply the variance of the simulated population and is the square of the standard deviation of $F$ reported in the third column. The percentage bias of $\bar{V}$ was also computed with respect to the theoretical asymptotic variance and in almost every case was smaller than the biases reported in this column. For purposes of inference, the bias with respect to the simulated population variance is more relevant.
* The $\chi^2$-test was performed using 50 equiprobable categories yielding 49 degrees of freedom. A p-value of 0.01 indicates that the actual p-value is less than 0.005.
* For 1000 replications, the tabulated 90%- and 95%-confidence intervals (equal weights in each tail) for the sample skewness statistic are [-0.127, 0.127] and [-0.180, 0.180], respectively [see Pearson and Hartley (1970)].
* For 1000 replications, the tabulated 90%- and 95%-confidence intervals (equal weights in each tail) for the sample studentized range statistic are [5.79, 7.33] and [5.68, 7.54], respectively [see Fama (1976)].
Table 5a

Means ($\bar{F}$, $\bar{V}_f$), standard deviations, and percentage biases for the sampling distribution of the call-option-price estimator ($\hat{F}$) and its asymptotic variance estimator ($V_f$) and normality tests for the sampling distribution of the standardized call-option-price estimators ($z$) for deep in- and out-of-the-money options: exercise prices $E = 25$ and $55$ for stock price $S = 40$, and fixed time-to-maturity $\tau = 1$ (week). For each option, three experiments were performed corresponding to sample sizes of 100, 300, 500, respectively. All experiments involve 1000 replications and assume an (annual) interest rate of 10% (yielding a continuously-compounded weekly rate of 0.183%) and a (weekly) variance rate of 1%.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$F$ (theoretical value)</th>
<th>$\bar{F}$ (std. dev.)</th>
<th>% bias ($\bar{F}$)</th>
<th>$V_f$ (population value)</th>
<th>$\bar{V}_f$ (std. dev.)</th>
<th>% bias ($\bar{V}_f$)</th>
<th>$z$</th>
<th>$\chi^2$-test</th>
<th>Skewness</th>
<th>Studentized range</th>
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<td>$E = 25$</td>
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<tr>
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<td>15.0458</td>
<td>15.0458</td>
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<td>4.0050 x 10^-11</td>
<td>-117.36</td>
<td>-13.2558</td>
<td>2525.3</td>
<td>0.99</td>
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<td></td>
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<td>(3.17 x 10^-10)</td>
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<td>(0.00)</td>
<td></td>
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<td>15.0458</td>
<td>0.00</td>
<td>2.3736 x 10^-12</td>
<td>-50.54</td>
<td>-0.7866</td>
<td>1055.7</td>
<td>0.99</td>
<td>-7.566</td>
<td>14.71</td>
</tr>
<tr>
<td></td>
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<td>(6.24 x 10^-13)</td>
<td></td>
<td></td>
<td>(3.07)</td>
<td>(0.00)</td>
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<tr>
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<td>7.9271 x 10^-13</td>
<td>-36.56</td>
<td>-0.5064</td>
<td>650.3</td>
<td>0.99</td>
<td>-3.431</td>
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<tr>
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<td>(7.62 x 10^-7)</td>
<td>(1.39 x 10^-12)</td>
<td></td>
<td></td>
<td>(1.70)</td>
<td>(0.00)</td>
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<tr>
<td>$E = 55$</td>
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<td></td>
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<tr>
<td>100</td>
<td>0.0010</td>
<td>0.0013</td>
<td>-31.41</td>
<td>1.7998 x 10^-6</td>
<td>-34.26</td>
<td>-0.6481</td>
<td>744.5</td>
<td>0.99</td>
<td>-5.039</td>
<td>14.24</td>
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<tr>
<td></td>
<td>(1.16 x 10^-3)</td>
<td>(3.16 x 10^-6)</td>
<td></td>
<td></td>
<td>(2.26)</td>
<td>(0.00)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>0.0010</td>
<td>0.0011</td>
<td>-8.37</td>
<td>3.4515 x 10^-7</td>
<td>-11.51</td>
<td>-0.3629</td>
<td>416.4</td>
<td>0.99</td>
<td>-2.167</td>
<td>7.91</td>
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<tr>
<td></td>
<td>(5.56 x 10^-4)</td>
<td>(3.27 x 10^-7)</td>
<td></td>
<td></td>
<td>(1.40)</td>
<td>(0.00)</td>
<td></td>
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</tr>
<tr>
<td>500</td>
<td>0.0010</td>
<td>0.0011</td>
<td>-9.52</td>
<td>2.0034 x 10^-7</td>
<td>1.60</td>
<td>0.1458</td>
<td>191.9</td>
<td>0.99</td>
<td>-0.995</td>
<td>6.78</td>
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<td></td>
<td>(4.51 x 10^-4)</td>
<td>(1.53 x 10^-7)</td>
<td></td>
<td></td>
<td>(1.12)</td>
<td>(0.00)</td>
<td></td>
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</tr>
</tbody>
</table>

*The percentage bias is computed as $100 \times (true\ F - \bar{F})/true\ F$. A value of 0.00 indicates that the actual percentage bias is less than 0.005 percent.

*The percentage bias is computed as $100 \times (population\ V_f - \bar{V}_f)/population\ V_f$. Note that the 'population variance' is simply the variance of the simulated population and is the square of the standard deviation of $\bar{F}$ reported in the third column. The percentage bias of $\bar{V}_f$ was also computed with respect to the theoretical asymptotic variance and in almost every case was smaller than the biases reported in this column. For purposes of inference, the bias with respect to the simulated population variance is more relevant.

*The $\chi^2$-test was performed using 50 equiprobable categories yielding 49 degrees of freedom. A $p$-value of 0.00 indicates that the actual $p$-value is less than 0.005.

*For 1000 replications, the tabulated 90% and 95% confidence intervals (equal weights in each tail) for the sample skewness statistic are [-0.127, 0.127] and [-0.180, 0.180], respectively [see Pearson and Hartley (1970)].

*For 1000 replications, the tabulated 90% and 95% confidence intervals (equal weights in each tail) for the sample studentized range statistic are [5.79, 7.33] and [5.68, 7.54], respectively [see Fama (1976)].
Table 5b

Means ($\bar{F}, \bar{V}_F$), standard deviations, and percentage biases for the sampling distribution of the call-option-price estimator ($\hat{F}$) and its asymptotic variance estimator ($\hat{V}_F$) and normality tests for the sampling distribution of the standardized call-option-price estimators ($z$) for deep in- and out-of-the-money options: exercise prices $E = 25$ and $55$ for stock price $S = 40$, and fixed time-to-maturity $T = 13$ (weeks). For each option, three experiments were performed corresponding to sample sizes of 100, 300, 500, respectively. All experiments involve 1000 replications and assume an (annual) interest rate of 10% (yielding a continuously-compounded weekly rate of 0.183%) and a (weekly) variance rate of 1%.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$F$ (theoretical value)</th>
<th>$\bar{F}$ (std. dev.)</th>
<th>$V_F$ (population value)</th>
<th>$\bar{V}_F$ (std. dev.)</th>
<th>$%$ bias ($\bar{F}$)$^a$</th>
<th>$%$ bias ($\bar{V}_F$)$^b$</th>
<th>$\bar{z}$ (std. dev.)</th>
<th>$\chi^2$-test$^c$ (p-value)</th>
<th>Skewness$^d$</th>
<th>Studentized range$^e$</th>
</tr>
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<tr>
<td>$E = 25$</td>
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<td></td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>100</td>
<td>16.0252</td>
<td>16.0250</td>
<td>$1.5241 \times 10^{-2}$</td>
<td>$1.5292 \times 10^{-2}$</td>
<td>$-0.33$</td>
<td>$-0.2230$</td>
<td>135.3</td>
<td>0.926</td>
<td>7.76</td>
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</tr>
<tr>
<td></td>
<td>(0.123)</td>
<td></td>
<td>($6.19 \times 10^{-3}$)</td>
<td>($6.77 \times 10^{-3}$)</td>
<td></td>
<td></td>
<td>(1.12)</td>
<td>(0.00)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>16.0252</td>
<td>16.0271</td>
<td>$4.5810 \times 10^{-3}$</td>
<td>$5.0580 \times 10^{-3}$</td>
<td>$-10.41$</td>
<td>$-0.0800$</td>
<td>71.5</td>
<td>0.409</td>
<td>7.21</td>
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</tr>
<tr>
<td></td>
<td>(6.77 $\times 10^{-2}$)</td>
<td></td>
<td>($1.12 \times 10^{-3}$)</td>
<td>($1.02$)</td>
<td></td>
<td></td>
<td>(0.98)</td>
<td>(0.02)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>16.0252</td>
<td>16.0244</td>
<td>$2.7636 \times 10^{-3}$</td>
<td>$3.0004 \times 10^{-3}$</td>
<td>$-8.57$</td>
<td>$-0.0997$</td>
<td>74.9</td>
<td>0.406</td>
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<tr>
<td></td>
<td>(5.26 $\times 10^{-2}$)</td>
<td></td>
<td>($5.18 \times 10^{-4}$)</td>
<td>($1.01$)</td>
<td></td>
<td></td>
<td>(0.99)</td>
<td>(0.01)</td>
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<tr>
<td>$E = 55$</td>
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</tr>
<tr>
<td>100</td>
<td>1.9301</td>
<td>1.9212</td>
<td>$1.1623 \times 10^{-1}$</td>
<td>$1.1040 \times 10^{-1}$</td>
<td>$5.02$</td>
<td>$-0.1578$</td>
<td>117.6</td>
<td>0.643</td>
<td>6.06</td>
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<td>($2.62 \times 10^{-2}$)</td>
<td>($2.10 \times 10^{-2}$)</td>
<td></td>
<td></td>
<td>(1.10)</td>
<td>(0.00)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>1.9301</td>
<td>1.9275</td>
<td>$3.9138 \times 10^{-2}$</td>
<td>$3.6794 \times 10^{-2}$</td>
<td>$5.99$</td>
<td>$-0.0862$</td>
<td>63.3</td>
<td>0.366</td>
<td>6.78</td>
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<tr>
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<td>(0.198)</td>
<td></td>
<td>($5.06 \times 10^{-3}$)</td>
<td>($5.06 \times 10^{-3}$)</td>
<td></td>
<td></td>
<td>(1.05)</td>
<td>(0.01)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>1.9301</td>
<td>1.9330</td>
<td>$2.1940 \times 10^{-2}$</td>
<td>$2.2139 \times 10^{-2}$</td>
<td>$0.91$</td>
<td>$-0.0322$</td>
<td>51.0</td>
<td>0.239</td>
<td>6.64</td>
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<td></td>
<td>(0.148)</td>
<td></td>
<td>($2.28 \times 10^{-3}$)</td>
<td>($2.28 \times 10^{-3}$)</td>
<td></td>
<td></td>
<td>(1.00)</td>
<td>(0.39)</td>
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</tbody>
</table>

$^a$The percentage bias is computed as $100 \times (\text{true } F - \hat{F})/(\text{true } F)$.

$^b$The percentage bias is computed as $100 \times (\text{population } V_F - \hat{V}_F)/(\text{population } V_F)$. Note that the 'population variance' is simply the variance of the simulated population and is the square of the standard deviation of $F$ reported in the third column. The percentage bias of $\hat{V}_F$ was also computed with respect to the theoretical asymptotic variance and in almost every case was smaller than the biases reported in this column. For purposes of inference, the bias with respect to the simulated population variance is more relevant.

$^c$The $\chi^2$-test was performed using 50 equiprobable categories yielding 49 degrees of freedom. A $p$-value of 0.00 indicates that the actual $p$-value is less than 0.005.

$^d$For 1000 replications, the tabulated 90%- and 98%-confidence intervals (equal weights in each tail) for the sample skewness statistic are $[-0.127, 0.127]$ and $[-0.180, 0.180]$, respectively [see Pearson and Hartley (1970)].

$^e$For 1000 replications, the tabulated 90%- and 95%-confidence intervals (equal weights in each tail) for the sample studentized range statistic are [5.79, 7.33] and [5.68, 7.54], respectively [see Fama (1976)].
increases, the decrease is not monotonic. This is to be expected since each experiment is random and independent of others and is subject to the usual sampling variation.

The biases for deep in- or out-of-the-money options, however, are quite large when the time-to-maturity is one week. Table 5a displays price biases of up to \(-31.41\%\) and asymptotic-variance biases of over \(-117\%\). This suggests that caution must be exercised in using these estimators for deep in- or out-of-the-money options just about to expire. However, the percentage bias is misleading in this case since the asymptotic variances are essentially zero and the estimator is virtually non-stochastic. Intuitively, this is simply due to the fact that the value of deep in- or out-of-the-money call options with short times-to-maturity do not depend significantly on the variance of the stock-price process if the process is a pure diffusion. Since the simulated stock prices are constructed as lognormal diffusions, it is not surprising that with one week to go, the estimator for deep in- and out-of-the-money options has little asymptotic variation. In this case, a minute absolute difference between the theoretical and finite-sample asymptotic variances can yield an extraordinary percentage bias.\(^{12}\) Table 5b shows that, as the time-to-maturity increases, the bias declines dramatically, the largest price bias being 0.46% and the largest variance bias being \(-10.41\%\).

3.3. Finite sample properties of the \(z\)-statistic

Consider now the asymptotic behavior of the statistic \(z\). Under the null hypothesis that \(z\) is standard normal, the \(\chi^2\)-test is performed for the 1000 replications of each experiment with 50 equiprobable categories yielding 49 degrees of freedom. From tables 4 and 5, it seems that with a sample size of 100 weekly observations for stock prices, the standard normality of \(z\) may be rejected at almost any level of significance. However, in most cases the null hypothesis of normality may be accepted at levels of 5% or smaller with 300 or more weekly observations of stock-price data. Nevertheless, it may be noted that the means of \(z\) are negative for almost all experiments. For the purpose of detecting skewness departures from normality, the skewness coefficient may yield a more powerful test than the \(\chi^2\)-test. Under the null hypothesis that \(z\) is standard normal, the distribution of the sample skewness coefficient has been tabulated\(^{13}\) and, for 1000 replications, the 90%-confidence interval is \([-0.127, 0.127]\) and the 98%-confidence interval is \([-0.180, 0.180]\). It is clear that even in cases where the \(\chi^2\)-test does not reject the null hypothesis of standard normality, the skewness coefficient is often outside the 98%-confidence interval. This indicates that the finite-sampling distribution of \(z\) is

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\(^{12}\)I am grateful to Jay Shanken for raising this point.

\(^{13}\)See Pearson and Hartley (1970, table 34B).
skewed (to the left). However, if the ‘tail behavior’ of $z$ is close to that of the standard normal, then the hypothesis tests based on $z$ suggested in section 2 are in fact appropriate. To measure possible departures from standard normality in the tails of the finite-sample distribution of $z$, the studentized range for each experiment may be compared with its tabulated distribution under the null hypothesis.\textsuperscript{14} For 1000 replications, the 90%-confidence interval of the studentized range with 5% in each tail is given by [5.79, 7.33] and the 95%-confidence interval with 2.5% in each tail is [5.68, 7.54]. For the hypothetical options in tables 4a and 4b, only in one case does the computed studentized range fall outside the 90%-confidence interval. This suggests that, although the finite-sample distribution of $z$ may be skewed, its tail probabilities match the standard normal’s fairly closely. For purposes of testing the BS model as specified by (1), the results seem to support the use of the $z$-statistic as described in the previous section for options not too deep in- or out-of-the-money. Table 5a shows that for deep in-the-money options with 1 week to go, not even a sample size of 500 is sufficient to produce the asymptotic results for $z$; both the studentized range and the $\chi^2$-tests reject normality at practically any level of significance. However, for deep out-of-the-money options with 1 week to go, sample sizes of 500 or more seem to be sufficient to render the tail behavior of $z$ close to the standard normal’s as measured by the studentized range. The results in table 5b however show that once the time-to-maturity increases to 13 weeks, the tail behavior of the $z$-statistic matches that of the standard normal even for deep in- or out-of-money options.

Although the simulation results reported above correspond to experiments carried out at the weekly frequency, further sampling experiments were performed at the daily frequency, and the results were quite similar, i.e., skewed distributions with normal tails. Since Boyle and Ananthanarayanan (1977) have already established that the means of the option-price estimators are acceptably close to the true values, what is at issue in these sampling experiments is the variability of the estimators (specifically the $z$-statistic) as the sample size grows. It is therefore not surprising that 300 weekly observations yield sampling properties similar to that of 300 daily observations since, loosely speaking, for purposes of estimating variances it is \textit{ceteris paribus} the frequency of observation that affects efficiency and not the calendar length of observation.

From the simulation evidence provided above, it may be concluded that if the call-option-pricing model (1) obtains, then for options which are not too deep in- or out-of-the-money and for deep in- or out-of-the-money options which are not just about to expire, using its asymptotic distribution for purposes of testing and inference may well be justified.

\textsuperscript{14}See Fama (1976, table 1.9, p. 40).
4. The general methodology

Although the above analysis involved the BS option-pricing model, this section demonstrates that the previously outlined methodology may be applied to virtually all contingent-claims models. In fact, it will become clear that even if the contingent-claims pricing function cannot be derived in closed form, its asymptotic distribution is still normal with a limiting variance which may be estimated numerically.

4.1. Estimation of Itô processes

Since almost all contingent claims models assume that fundamental asset prices follow Itô processes, it is first necessary to consider the estimation problem for this class of stochastic processes. For expositional clarity we only consider the estimation problem for Itô processes with single jump and diffusion components. The extension to multiple jump and diffusion terms and vector Itô processes poses no conceptual difficulties, but is notationally more cumbersome. Let $X(t)$ be an Itô process with domain $\Omega \subset R$ satisfying the following stochastic differential equation:

$$dX = f(X, t; \alpha) \, dt + g(X, t; \beta) \, dW + h(X, t; \gamma) \, dN_\lambda, \quad t \in [0, \infty),$$ (11)

where $dW$ is the standardized Wiener process, and $dN_\lambda$ is a Poisson counter (jump magnitude = 1), independent of $dW$, with intensity $\lambda$. There is clearly no loss of generality in assuming that the jump magnitude is unity since this is merely a normalization which may be subsumed by the coefficient function $h$.

In addition to assuming those conditions which insure the existence and uniqueness of the solution to (11), we make the following additional assumptions:

(A.1) Coefficient functions $f$, $g$, and $h$ are known up to parameter vectors $\alpha, \beta, \gamma, \lambda$, respectively. The true but unknown parameters $\alpha_0, \beta_0, \gamma_0, \lambda_0$

15Suppose, however, the jump magnitude is stochastic. More generally, suppose that certain "parameters" in $f$, $g$, and $h$ are in fact random variables. Without further information, there is of course little that can be done. If, however, it is posited that these random parameters are distributed according to a particular parametrizable probability law which is statistically independent of $dW$ and $dN_\lambda$, then the estimation procedure described in this section may still be applied. For example, if it is assumed that the jump magnitude is lognormally distributed with unknown parameters and is independent of $dW$ and $dN_\lambda$, these parameters may be estimated along with the other unknown parameters of $f$, $g$, and $h$ as well.

16See Arnold (1974, ch. 6).
lie in the interior of the compact parameter spaces \( A, B, \Gamma, \Lambda \), respectively. Let \( \theta_0 = (\alpha_0, \beta_0, \gamma_0, \lambda_0)' \) and let \( \Theta = A \times B \times \Gamma \times \Lambda \). The functions \( f, g, \) and \( h \) are twice continuously differentiable in \((X, t)\) and three times continuously differentiable in \( \theta \).

(A.2) \( n \) observations of \( X(t) \) are taken at times \( t_1, t_2, \ldots, t_n \) not necessarily equally spaced apart, where \( 0 < t_1 < \cdots < t_n \). \( X \equiv (X_1, X_2, \ldots, X_n)' \), where \( X_i = X(t_i) \), \( i = 1, \ldots, n \). \( X(t_0) = X_0 \) is known.

We may now state the estimation problem as: Given the observations \( X \) and the process dynamics (11), find the optimal estimator \( \hat{\theta} \) of the true parameters \( \theta_0 \). By restricting consideration to the class of consistent and uniformly asymptotically normal (CUAN) estimators, it has been shown that the ML estimator is optimal in the sense that it has the smallest asymptotic variance of all other CUAN estimators. For this reason, ML estimation is the preferred approach. The ML estimator is obtained by considering the joint density function of the random sample \( X \) as a function of the unknown parameters, and then finding that value \( \hat{\theta}_{\text{ML}} \) which maximizes the joint density in \( \theta \). We now proceed to derive this joint density function which, when considered a function of the parameters \( \theta \) given the data \( X \), is called the joint-likelihood function.

Let \( \rho(X_1, \ldots, X_n|X_0) \) denote the joint-density function of the random sample \( X \), where the dependence of \( \rho \) on the unknown parameters \( \theta \) and on \( t_1, \ldots, t_n \) has been suppressed for notational simplicity. The density \( \rho \) may always be written as the following product of conditional densities:

\[
\rho(X_1, \ldots, X_n|X_0) = \rho_1(X_1|X_0)\rho_2(X_2|X_1, X_0) \cdots \rho_n(X_n|X_{n-1}, \ldots, X_0).
\]

However, since \( X(t) \) is a Markov process\(^{17}\) eq. (12) reduces to

\[
\rho(X_1, \ldots, X_n|X_0) = \rho_1(X_1|X_0)\rho_2(X_2|X_1)\rho_3(X_3|X_2) \cdots \rho_n(X_n|X_{n-1}).
\]

For compactness of notation, we will write \( \rho_k(X_k, t_k|X_{k-1}, t_{k-1}) \) as \( \rho_k \).

Given the functions \( f, g, \) and \( h \), the joint density function \( \rho(X) \) of the random sample \( X \) may be derived by solving the Fokker–Planck or forward equation for the transition densities \( \rho_k \) subject to any boundary conditions which may apply. Although the resulting functional partial differential equa-

\(^{17}\)See Kushner (1967).
tion characterizes the transition densities\(^{18}\) (hence the conditional likelihood functions), obtaining a closed-form solution for the \(\rho_k\)'s is generally quite difficult. However, by restricting the functional forms of \(f\), \(g\), and \(h\), it is often possible to derive the transition densities explicitly. For example, if \(h = 0\) (pure diffusion) and \(f\) and \(g\) satisfy a certain 'reducibility' condition, it may be shown\(^{19}\) that there exists a transformed process \(Z(t)\) of \(X(t)\) for which the coefficient functions are independent of \(Z(t)\). That is, for some suitable change of variables \(T[X(t)] = Z(t)\), an application of Itô's lemma will yield

\[
dZ = p(t; \theta) \, dt + q(t; \theta) \, dW. \tag{14}
\]

In this case the transition density for the transformed data is readily derived as

\[
\rho_k(Z, t) = \left[2\pi \int_{t_{k-1}}^{t} q^2 \, d\tau \right]^{-\frac{1}{2}} \exp \left[ -\frac{(Z - Z_{k-1} - \int_{t_{k-1}}^{t} p \, d\tau)^2}{2 \int_{t_{k-1}}^{t} q^2 \, d\tau} \right]. \tag{15}
\]

The well-known lognormal diffusion process (2) is an example, for which the transformation \(T(x)\) is just \(\ln X\) and \(p = \mu - \frac{1}{2} \sigma^2\), \(q = \sigma\).

Given the transition densities \(\rho_k\), the joint-likelihood and log-likelihood functions of the random sample \(X\) are given by

\[
L(\theta; X) = \prod_{k=1}^{n} \rho_k(X_k, t_k|X_{k-1}, t_{k-1}; \theta), \tag{16a}
\]

\[
G(\theta; X) = \sum_{k=1}^{n} \ln \rho_k(X_k, t_k|X_{k-1}, t_{k-1}; \theta)
\equiv \sum_{k=1}^{n} l_k(X_k|X_{k-1}; \theta), \tag{16b}
\]

\(^{18}\)Specifically, the conditional likelihood function is the solution to the following functional partial differential equation:

\[
\frac{\partial}{\partial t} [\rho_k] = - \frac{\partial}{\partial \mathbf{x}} [f \rho_k] + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{x}^2} (g^2 \rho_k) - \lambda \rho_k + \lambda \tilde{\rho}_k \left| \frac{\partial}{\partial \mathbf{x}} [h^{-1}] \right|.
\]

where

\[
\tilde{h}(X, t; \gamma) = X + h(X, t; \gamma), \quad \tilde{h}(X^{-1}, t; \gamma) = X,
\]

\[
\tilde{\rho}_k = \rho_k(\tilde{h}^{-1}, t).
\]

\[
\rho_k(X, t_{k-1}|X_{k-1}, t_{k-1}) = \delta(X - X_{k-1}),
\]

and \(\delta(X - X_{k-1})\) is the Dirac-delta generalized function centered at \(X_{k-1}\). See Lo (1985) for its derivation and a more complete discussion of its solutions and the general estimation problem.

\(^{19}\)See Schuss (1980, ch. 4) for a statement of the reducibility condition.
where the dependence of \( l_k \) on time has been suppressed for notational simplicity. Under assumptions (A.1) and (A.2) and mild regularity conditions, the ML estimator \( \hat{\theta}_{ML} \) of \( \theta_0 \) exists, is consistent, and is asymptotically efficient in the class of all CUAN estimators. That is,

\[
\lim_{n \to \infty} \frac{\sqrt{n}}{\sum_j \left( \hat{\theta}_{ML} - \theta_0 \right) \sim N(0, I^{-1}(\theta_0)).}
\]

where the asymptotic covariance matrix \( I^{-1}(\theta_0) \) is the inverse of the information matrix \( I(\theta_0) \):

\[
I(\theta_0) = \lim_{n \to \infty} \frac{\sum_k \left( \frac{\partial^2 l(X_k|X_{k-1}; \theta_0)}{\partial \theta \partial \theta'} \right)}{n}
\]

4.2. Two examples

For illustrative purposes, the likelihood functions of two particular processes are presented below.

**Example 1. Ornstein–Uhlenbeck process**

As an illustration of a general equilibrium characterization of the term structure of interest rates, Vasicek (1977) considers the specific process

\[
dX = \alpha(y - X) dt + \beta dW,
\]

which is an Ornstein–Uhlenbeck process with steady-state mean \( y \). Because the increments of such a process are normally distributed, the conditional likelihood is particularly easy to derive and is given by

\[
\rho(X_k, t_k|X_{k-1}, t_{k-1})
\]

\[
= \left[ \frac{\pi \beta^2}{\alpha} (1 - e^{-\alpha \Delta t_k}) \right]^{-\frac{1}{2}} \exp \left[ -\frac{\alpha(X_k - X_{k-1}e^{-\alpha \Delta t_k} + y[1 - e^{\alpha \Delta t_k}])^2}{\beta^2(1 - e^{-2\alpha \Delta t_k})} \right].
\]

**Example 2. Diffusion with absorbing barrier**

Although Black and Cox (1976) and Ho and Singer (1982) have derived valuation formulas for risky debt with various indenture provisions, to date

\[\text{For perhaps its first application in the econometrics literature, see Hausman and Wise (1983).} \]
they have not been empirically implemented. As a simple example of how corporate bankruptcy might be modelled and estimated, let \( X(t) \) represent a firm’s equity price at time \( t \) and suppose \( X(t_0) > 0 \) and that \( X(t) \) follows arithmetic Brownian motion:

\[
dX = \alpha dt + \beta dW. \tag{21}
\]

Furthermore, let \( X = 0 \) be an absorbing state so that, if \( X(t) \) reaches 0, it stays in that state thereafter, i.e., bankruptcy occurs. Now suppose that \( n \) observations of \( X \) are taken and that \( X_1 > 0, \ldots, X_{n-1} > 0, X_n = 0 \) so that bankruptcy occurs in this sample some time between \( t_{n-1} \) and \( t_n \). Then the likelihood function for this sample would be the product of the conditional densities for observations \( X_1 \) to \( X_{n-1} \) where

\[
p(X_k, t_k | X_{k-1}, t_{k-1}) = \left[ 2\pi \beta^2 \Delta t_k \right]^{-\frac{1}{2}} \exp \left\{ -\frac{(X_k - X_{k-1} - \alpha \Delta t_k)^2}{2\beta^2 \Delta t_k} \right\},
\]

\[ k = 1, \ldots, n - 1, \tag{22} \]

multiplied by the distribution function of the first-passage time for observation \( X_n \). Following Cox and Miller’s (1973) derivation for the first-passage time distribution of a process with an absorbing barrier at \( X = a > 0 \), the distribution for the barrier at \( X = 0 \) may be calculated to be

\[
P(\text{Absorption in } [t_{n-1}, t_n]) = \Phi\left[ \frac{-X_{n-1} - \alpha \Delta t_n}{\beta \sqrt{\Delta t_n}} \right] + \exp\left[ -\frac{2\alpha X_{n-1}}{\beta^2} \right] \Phi\left[ \frac{-X_{n-1} + \alpha \Delta t_n}{\beta \sqrt{\Delta t_n}} \right], \tag{23} \]

where \( \Phi \) is the standard normal distribution function. Note that although \( X(t) \) may have been absorbed at any time between \( t_{n-1} \) and \( t_n \), knowing that \( X(t) \) has been absorbed by time \( t_n \) is sufficient for computing ML estimates of the unknown parameters. Given ML estimates of \( \alpha \) and \( \beta \), it is then possible to obtain estimates of the probability of bankruptcy within any given time interval for firms which are similar to the one which generated the original sample \( X \). For example, it may be plausible to make inferences about the probability of default for a specific small savings and loan association using estimates based on data for a representative cross-section of small banks. Of course, the reliability of such bankruptcy forecasts depends on the relative impact of industry effects versus idiosyncratic effects in triggering default and is an empirical question.
4.3. The asymptotic distribution of general contingent-claims estimators

Let $F$ be the price of an arbitrary asset which is contingent upon the fundamental asset $X(t)$. In particular, suppose for now that $F$ may be determined by the following known asset-pricing formula:

$$F = F(X, t, \eta; \theta_0),$$

(24)

with $F$ continuously differentiable in $\theta$, where $\eta$ is a vector of observables (e.g., interest rates, time-to-maturity, etc.) and $\theta_0$ is the unknown true parameter vector associated with the fundamental asset-price process $X(t)$.

Given assumptions (A.1) and (A.2), the well-known 'principle of invariance' states that the ML estimator of the contingent-claims price $F$ is simply

$$\hat{F}_{ML} = F(X, t, \eta; \hat{\theta}_{ML}),$$

(25)

where $\hat{\theta}_{ML}$ maximizes (16). Since $\hat{F}_{ML}$ is a true ML estimator of $F$, it is also consistent and asymptotically efficient in the class of all CUAN estimators of $F$. In addition, the asymptotic distribution of the estimator $F_{ML}$ may be easily derived and is given by

$$\sqrt{n} \left( \hat{F}_{ML} - F \right) \overset{d}{=} N(0, V_0),$$

(26a)

$$V_0 = \frac{\partial F(\theta_0)^T}{\partial \theta} \hat{\theta}_{ML} - I^{-1}(\theta_0) \frac{\partial F(\theta_0)}{\partial \theta}. $$

(26b)

Using (26), the usual forms of statistical inference may then be applied to the estimated contingent-claims price. In particular, the model-specification test, confidence intervals, price forecast, and other forms of statistical inference which were suggested in section 2 for the BS call-option-pricing model may also be applied to any other type of contingent-claims model in similar fashion. In fact, even if it is not possible to solve in closed form the partial differential equation which yields the contingent-claims-pricing formula $F$, it may still be possible to calculate $\hat{F}_{ML}$ and its derivatives $\partial F(\hat{\theta}_{ML})/\partial \theta$ numerically. The asymptotic distribution is then completely determined, and the usual forms of statistical inference once again obtain. This is an example of a situation in which large-sample theory is the only practical form of statistical inference.

Specifically, if the solution to the fundamental partial differential equation determining $F$ exists, the ML estimate $\hat{F}_{ML}$ is obtained by numerically solving the p.d.e. with estimates of the parameters substituted into its coefficients as, for example, in Brennan and Schwartz (1980, 1982). The derivative $\partial F(\hat{\theta}_{ML})/\partial \theta$ may be evaluated by perturbing the value of $\hat{\theta}_{ML}$ slightly, re-solving for $\hat{F}_{ML}$ numerically, and then computing the ratio of the change in $F_{ML}$, i.e., $\partial F/\partial \theta = \Delta F/\Delta \theta$. The asymptotic variance is then readily computed. I am grateful to an anonymous referee for suggesting that this issue be explored.
inference available since, in this case, attempting to deduce the finite-sample
distribution of the contingent-claims estimator along the lines of Boyle and
Ananthanarayanan (1977) would be prohibitively expensive.

4.4. Implicit parameter 'tests' of contingent-claims models

One further issue in the parametric statistical inference problem for contin-
gent-claims models is whether or not such models may be tested by comparing
the implied parameter values of several contingent-claims on the same funda-
mental asset. As an example, consider two call options with different exercise
prices for a single stock. If the BS option-pricing model is strictly true, then
the implicit variances computed from the two options should be numerically
identical. Because this relationship must hold exactly under the BS model, the
statistical inference problem is rendered trivial in this case; such a test will
almost always reject the model. However, one of the assumptions in the BS
framework is that all investors know the true variance \( \sigma^2 \) without error, hence
the premise of the BS model eliminates the need for statistical inference
altogether. Nevertheless, it has been well-documented that in fact options on
the same stock do not yield identical implicit variances. That the BS model in
its simplest form is not completely consistent with actual market conditions is
not surprising. After all, one of the consequences of the BS approach is that
the options are purely redundant securities, which contradicts the very ex-
istence of large organized options markets. Indeed several of the assumptions
which Black and Scholes require (e.g., continuous trading, no transactions
costs, constant interest rate, lognormal diffusion of stock prices) may in fact be
violated in practice, leading to empirical inconsistencies such as the general
disagreement of implicit variances of options on a given stock.

Of course, there may be many alternate explanations as to why a contingent
claim's market price deviates from its theoretical price. Although valuation
formulas which are consistent with the data might exist even when the
standard assumptions are relaxed, they must clearly be re-derived from first
principles. For example, consider relaxing the assumption of the BS model
that \( \sigma^2 \) is known and suppose instead that investors estimate it consistently
(but with error). Under this assumption, the BS model no longer obtains. To
see this, consider the usual hedging argument of Black and Scholes where the
option is perfectly hedged at each point in time by continuously adjusting a
portfolio of the stock and a riskless bond. If the true variance is unknown,
then the perfect hedge cannot be constructed since the portfolio weights
depend upon \( \sigma^2 \). Furthermore, investors may have different information sets
with which they form their estimates of \( \sigma^2 \). However, it may not be unreason-
able to argue [as in Merton (1976) and Boyle and Emanuel (1980)] that the
residual risk in such a portfolio will be uncorrelated with the market return
and, hence, not priced. In this case, the expected return from such a portfolio
must also satisfy the usual BS partial differential equation from which the well-known valuation formula may be derived. Note that the implications of this approach differ from those of the standard BS model in several ways. First, the BS formula holds in this case for an economy where agents are well-diversified. If, for example, there were several large and undiversified investors, the unsystematic risk of the BS hedging portfolio may in fact be priced, causing the market price to deviate from the theoretical BS price. Also, the way in which the unsystematic risk affects investors clearly depends upon the stock price, exercise price, and time-to-maturity, as well as the characteristics of the investors. This implies that options on a given stock with, say, different exercise prices may deviate from the BS price to differing degrees depending upon the relative size and diversification of portfolios holding each issue of options. If this is the case, then the implied variances of options on the same stock need not be numerically identical. Furthermore, because investors need not base their variance estimates upon the same information set there may be disagreement concerning the value of an option, in which case trade will occur, i.e., an options market will arise. However, in this case the market price of options will also deviate from the BS price. Although this richer setting may allow many different sources of deviation, in this paper we focus only on one such source: general statistical fluctuations arising from estimating the underlying parameters.

5. Conclusion

In this paper we have provided a general methodology for the estimation and testing of general contingent-claims asset-pricing models by appealing to asymptotic statistical theory. Given the large-sample distribution of any contingent-claims-price estimator, the financial economist may bring to bear a considerable collection of statistical tools upon a variety of problems in model-specification testing and forecasting. Of course, exact small-sample properties are always preferable to their large-sample counterparts when available. However, due to the non-linear nature of most contingent-claims-price estimators, their exact distributions usually do not exist in closed form. The use of asymptotic distributions as approximations is a natural alternative. Moreover, because financial econometricians have at their disposal what most economists would consider 'large samples', applications of asymptotic approximations may yield quite accurate results. Since what constitutes a 'large sample' depends upon the particular estimator of interest, Monte Carlo studies must be performed on a case-by-case basis in order to determine the practical relevance of the proposed methods. The simulation results reported in section 3 for the Black–Scholes call-option-pricing model suggest that for most call options, a large sample consists of between 300 and 500 observations. More-
over, the costs of performing these simulation studies are quite small, certainly relative to their payoff but also in absolute magnitude. As an example, the costs of performing the simulations in tables 4 and 5 did not exceed $25.00.

In addition to cost effectiveness, another advantage of such large-sample results is tractability. The numerical estimation of the fundamental asset's parameters is a straightforward application of now standard maximum-likelihood software packages. In addition, part of the standard output of such packages is a consistent estimate of the inverse of the information matrix $I^{-1}$. Given this estimate, the asymptotic distribution of any corresponding contingent claim may then be derived by computing the derivative of its pricing formula with respect to the unknown parameters. For those contingent claims with tractable pricing formulas, expressions for their asymptotic distributions will also be tractable. The applicability of the proposed methods thus extends to virtually all contingent-claims models which are of theoretical interest since those are often ones for which pricing formulas may be determined either explicitly or numerically. Although this approach seems quite promising, whether or not the application of these results to other contingent-claims models will yield new insights can only be determined by further empirical investigations.

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