Asset Prices and Trading Volume under Fixed Transactions Costs

Andrew W. Lo

Massachusetts Institute of Technology and National Bureau of Economic Research

Harry Mamaysky

Morgan Stanley and Yale University

Jiang Wang

Massachusetts Institute of Technology, China Center for Financial Research, and National Bureau of Economic Research

We propose a dynamic equilibrium model of asset prices and trading volume when agents face fixed transactions costs. We show that even small fixed costs can give rise to large "no-trade" regions for each agent's optimal trading policy. The inability to trade more frequently reduces the agents' asset demand and in equilibrium gives rise to a significant illiquidity discount in asset prices.

I. Introduction

It is now well established that transactions costs in asset markets are an important factor in determining the trading behavior of market partic-

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ipants.[1] Consequently, transactions costs should also affect market liquidity and asset prices in equilibrium.[2] However, the direction and magnitude of their effects on asset prices, trading volume, and other market variables are still subject to considerable controversy and debate.

Early studies of transactions costs in asset markets relied primarily on partial equilibrium analysis. For example, by comparing exogenously specified returns of two assets—one with transactions costs and another without—that yield the same utility, Constantinides (1986) argued that proportional transactions costs have only a small impact on asset prices. However, using the present value of transactions costs under a set of candidate trading policies as a measure of the liquidity discount in asset prices, Amihud and Mendelson (1986*b*) concluded that the liquidity discount can be substantial, despite relatively small transactions costs.

More recently, several authors have developed equilibrium models to address this issue. For example, Heaton and Lucas (1996) numerically solve a model in which agents trade to share their labor income risk and conclude that symmetric transactions costs alone do not affect asset prices significantly. Vayanos (1998) develops a model in which agents trade to smooth lifetime consumption and shows that the price impact of proportional transactions costs is linear in the costs and that for empirically plausible magnitudes their impact is small. Huang (2003) considers agents who are exposed to surprise liquidity shocks and who are able to trade in a liquid and an illiquid financial asset. He also finds that in the absence of additional constraints, the liquidity premium is small.

A common feature of these equilibrium models is that agents have only infrequent trading needs. Such models may understate the effect of transactions costs on asset prices, given the much higher levels of trading activity that we observe empirically. This suggests the need for a more plausible model of trading behavior to fully capture the economic implications of transactions costs in financial markets.

In this paper, we provide such a model by investigating the impact of fixed transactions costs on asset prices and trading behavior in a continuous-time equilibrium model with heterogeneous agents. Investors are endowed with a nontradable risky asset, and in a frictionless

[1] The literature on optimal trading policies in the presence of transactions costs is vast (see, e.g., Constantinides 1976, 1986; Eastham and Hastings 1988; Davis and Norman 1990; Dumas and Luciano 1991; Morton and Pliska 1995; Schroeder 1998). The impact of transactions costs on agents' economic behavior has been studied in many other contexts as well (see, e.g., Baumol 1952; Tobin 1956; Arrow 1968; Rothschild 1971; Bernanke 1985; Pindyck 1988; Dixit 1989).

[2] See, e.g., Demsetz (1968), Garman and Ohlson (1981), Amihud and Mendelson (1986*a*, 1986*b*), Grossman and Laroque (1990), Aiyagari and Gertler (1991), Dumas (1992), Tuckman and Vila (1992), Heaton and Lucas (1996), Vayanos (1998), Vayanos and Vila (1999), and Huang (2003).

economy they wish to trade continuously in the market, in amounts that are cumulatively unbounded, to hedge their nontraded risk exposure. But in the presence of a fixed transactions cost, they choose to trade only infrequently. Indeed, we find that even small fixed costs can give rise to large "no-trade" regions for each agent's optimal trading policy. Moreover, the uncertainty regarding the optimality of the agents' asset positions between trades reduces their asset demand, leading to a decrease in the asset price in equilibrium. We show that this price decrease—an "illiquidity discount"—satisfies a power law with respect to the fixed cost; that is, it is approximately proportional to the square root of the fixed cost, implying that small fixed costs can have a significant impact on asset prices. Moreover, the size of the illiquidity discount increases with the agents' trading needs at high frequencies and is very sensitive to their risk aversion.

Our model also allows us to examine how transactions costs can influence the level of trading volume. The apparently high level of volume in financial markets has often been considered puzzling from a rational asset-pricing perspective (see, e.g., Ross 1989), and some have even argued that additional trading frictions or "sand in the gears," in the form of a transactions tax, ought to be introduced to actively discourage higher-frequency trading (see, e.g., Tobin 1984; Stiglitz 1989; Summers and Summers 1990). Yet in the absence of transactions costs, most dynamic equilibrium models will show that it is quite rational and efficient for trading volume to be *infinite* when the information flow to the market is continuous, for example, a diffusion. An equilibrium model with fixed transactions costs can reconcile these two disparate views of trading volume. In particular, our analysis shows that while fixed costs do imply less than continuous trading and finite trading volume, an increase in such costs has only a slight effect on volume at the margin.

We develop the basic structure of our model in Section II and discuss the nature of market equilibrium under fixed transactions costs in Section III. We derive an explicit solution for the dynamic equilibrium in Section IV and analyze the solution in Section V. Section VI reports the results of a calibration exercise of our model, and we present conclusions in Section VII. Proofs appear in the Appendix in the online edition of the article.

II. The Model

Our model consists of a continuous-time dynamic equilibrium in which heterogeneous agents trade with each other over time to hedge their exposure to nontraded risk. Our interest in the trading process requires that we consider more than one agent, and because we seek to capture both the *time of trade* and the quantity of trade in an equilibrium setting, we develop our model in continuous time. However, for tractability and economic clarity, we maintain parsimony in modeling the heterogeneity among agents, their trading motives, and the economic environment.

A. *The Economy*

Our economy is defined over a continuous-time horizon $[0, \infty)$ and contains a single commodity that is used as the numeraire. The underlying uncertainty of the economy is characterized by a two-dimensional standard Brownian motion $B = \{(B_{1t}, B_{2t}) : t \geq 0\}$ defined on its filtered probability space $(\Omega, \mathcal{F}, F, \mathcal{P})$, where the filtration $F = \{\mathcal{F}_t : t \geq 0\}$ represents the information revealed by B over time.

There are two traded securities: a risk-free bond and a risky stock. The bond pays a positive, constant interest rate r. Each stock share pays a cumulative dividend D_t, where

$$D_t = \bar{a}_D t + \int_0^t \sigma_D dB_{1s} = \bar{a}_D t + \sigma_D B_{1t} \tag{1}$$

and $\bar{a}_D$ and σ_D are positive constants. The securities are traded competitively in a securities market. Let $P = \{P_t : t \geq 0\}$ denote the stock price process.

Transactions in the bond market are costless, but transactions in the stock market are costly. For each stock transaction, the buyer and seller have to pay a combined fixed cost of κ that is exogenous and independent of the amount transacted. The *allocation* of this fixed cost between buyer and seller, denoted by κ^+ and κ^-, respectively, is determined endogenously in equilibrium. More formally, the transactions cost for a trade δ is given by

$$\kappa(\delta) = \begin{cases} \kappa^+ & \text{for } \delta > 0 \\ 0 & \text{for } \delta = 0 \\ \kappa^- & \text{for } \delta < 0, \end{cases} \tag{2}$$

where δ is the signed volume (positive for purchases and negative for sales), κ^+ is the cost to the buyer, κ^- is the cost to the seller, and the sum $\kappa^+ + \kappa^- = \kappa$ is fixed.

There are two agents in the economy, indexed by $i = 1, 2$, and each agent is initially endowed with no bonds and $\bar{\theta}$ shares of the stock. In addition, agent i is endowed with a stream of nontraded risky income with cumulative cash flow N_t^i, where

$$N_t^i = -\int_0^t (-1)^i X_s dB_{1s}, \tag{3a}$$

$$X_t = \sigma_X B_{2t}, \tag{3b}$$

and σ_X is a positive constant. For future reference, we let $X_t^i \equiv (-1)^i X_t$. The term B_{2t} specifies the nontraded risk, and X_t^i gives agent i's exposure to the nontraded risk at time t. Since $X_t^1 + X_t^2 = 0$ for all t, there is no aggregate nontraded risk. In addition, the nontraded risk is assumed to be perfectly correlated with stock dividends, allowing the agents to use the stock to hedge their nontraded risk. Since each agent's exposure to nontraded risk, X_t^i, is stochastic, he desires to trade in the stock market continuously to hedge his nontraded risk as it changes over time. The presence of this high-frequency trading need is essential in analyzing how transactions costs—which prevent the agents from trading continuously—affect their asset demands and equilibrium prices.

Each agent chooses his consumption and trading policy to maximize the expected utility from his lifetime consumption. Let C denote the agents' consumption space, which consists of F-adapted, integrable consumption processes $c = \{c_t : t \geq 0\}$. The agents' stock trading policy space consists of only "impulse" trading policies, defined as follows.

Definition 1. Let $\mathbb{N}_+ \equiv \{1, 2, \ldots\}$. An impulse trading policy $\{(\tau_k, \delta_k) : k \in \mathbb{N}_+\}$ is a sequence of trading times τ_k and trade amounts δ_k such that (1) $0 \leq \tau_k \leq \tau_{k+1}$ almost surely for all $k \in \mathbb{N}_+$, (2) τ_k is a stopping time with respect to F, (3) δ_k is measurable with respect to F_{τ_k}, (4) $\delta_k \leq \bar{\delta} < \infty$, and (5) $\mathrm{E}_0[e^{\gamma\kappa n(s)}] < \infty$, where

$$n(s) \equiv \sum_{\{k:\, 0 \leq \tau_k \leq s\}} 1$$

gives the number of trades in time $[0, s]$.

Conditions 1–3 are standard for impulse policies. Conditions 4 and 5 are imposed here for technical reasons. Condition 4 requires that trade sizes be finite.[3] Condition 5 requires that trading not be frequent enough to generate infinite trading costs. These are fairly weak conditions that any optimal policy should satisfy.

Agent i's stock holding at time t is θ_t^i, given by

$$\theta_t^i = \theta_{0^-}^i + \sum_{\{k:\, \tau_k^i \leq t\}} \delta_k^i, \tag{4}$$

where $\theta_{0^-}^i$ is his initial endowment of stock shares, which is assumed to be $\bar{\theta}$.

[3] Limiting trade sizes to be finite rules out potential doubling strategies. Effectively, we require the trading policy to be in the L^∞ space, which is a standard condition in continuous-time settings.

Let M_t^i denote agent i's bond position at t (in value). The term M_t^i represents agent i's liquid financial wealth. Then

$$M_t^i = \int_0^t (rM_s^i - c_s)ds + \int_0^t (\theta_s^i dD_s + dN_s^i) - \sum_{\{k:\, 0 \le \tau_k^i \le t\}} (P_{\tau_k^i}\delta_k^i + \kappa_k^i), \quad (5)$$

where $\kappa_k^i = \kappa(\delta_k^i)$ is given in (2). Equation (5) defines agent i's budget constraint. Agent i's consumption/trading policy (c, δ) is budget feasible if the associated M_t process satisfies (5).

Both agents are assumed to maximize expected utility of the form

$$u(c) = \mathrm{E}_0\left[-\int_0^\infty e^{-\rho t - \gamma c_t} dt\right], \quad (6)$$

where ρ and γ (both positive) are the time discount coefficient and the risk aversion coefficient, respectively. To prevent agents from implementing a "Ponzi scheme," we impose the following constraint on their policies for all $\gamma > 0$:

$$\mathrm{E}_0[-e^{-\rho t - r\gamma(M_t^i + \theta_t^i P_t) - r\gamma p^* X_t^i}] < \infty \quad \forall t \ge 0, \quad (7)$$

where p^* is an arbitrary positive number, representing the shadow price for future nontraded income.[4] The set of budget feasible policies that also satisfies the constraint (7) gives the set of admissible policies, which is denoted by Θ.

For the economy to be properly defined, we require the following condition:

$$4\gamma^2\sigma_X^2 \le 1, \quad (8)$$

which limits the volatility in each agent's exposure to the nontraded risk.

B. *Definition of Equilibrium*

DEFINITION 2. An equilibrium in the stock market is defined by

a. a price process $P = \{P_t : t \ge 0\}$ that is progressively measurable with respect to F,

b. an allocation of the transactions cost (κ^+, κ^-) between buyer and seller, and

[4] A more conventional constraint is imposed only on the terminal data. In the absence of nontraded income, the usual terminal condition is $\lim_{t\to\infty} \mathrm{E}_0[-e^{-\rho t - r\gamma W_t}] = 0$, where $W_t = M_t + \theta P_t$ is the terminal financial wealth. In this paper, for convenience, we impose a stronger condition (7), which limits agents from running an unbounded financial deficit at any point in time, not just in the limit.

c. agents' trading policies $\{(\tau_k^i, \delta_k^i) : k \in \mathbb{N}_+\}$, $i = 1, 2$, given the price process and the allocation of transactions costs,

such that

1. each agent's consumption/trading policy maximizes his lifetime expected utility:

$$J^i \equiv \sup_{(c,\delta)\in\Theta} \mathrm{E}\left[-\int_0^\infty e^{-\rho t-\gamma c_t^i}dt\right], \tag{9}$$

2. the stock market clears: for all $k \in \mathbb{N}_+$,

$$\tau_k^1 = \tau_k^2 \tag{10a}$$

and

$$\delta_k^1 = -\delta_k^2. \tag{10b}$$

C. *Discussion*

By assuming a constant interest rate, we are assuming that the bond market is exogenous. This assumption simplifies our analysis but deserves some discussion. In this paper we focus on how transactions costs affect the trading and pricing of a security when agents want to trade it at high frequencies. Assuming a constant interest rate allows us to focus on the stock, which is costly to trade, and to restrict our attention to simple risk-sharing motives for trading. Endogenizing the bond market would yield stochastic interest rates and introduce additional trading motives, for example, intertemporal hedging. While interesting in their own right, such complications are unnecessary for our current purposes.

For parsimony, we have also made several simplifying assumptions about the agents' nontraded risks, given in (3). First, we assume that there is no aggregate nontraded risk. In the current model, nontraded risk at the aggregate level does not generate any trading needs. It is the difference between agents in their nontraded risk that generates trading. Since we are mainly interested in the impact of transactions costs, it is natural to focus on the difference in nontraded risk across agents. After all, transactions costs matter only when agents want to transact. The difference in the agents' nontraded risk is fully characterized by X_t. We further assume that X_t follows a Brownian motion; hence changes in the difference between the agents' nontraded risk are persistent. In addition, we have assumed that the risk in the nontraded asset is instantaneously perfectly correlated with the risk of the stock. This implies that the nontraded risk is actually marketed. (Despite this, we continue to use the term "nontraded risk" throughout the paper to reflect the fact that it need not be marketed in general.) These two assumptions can potentially increase the agents' needs to trade; however, we do not expect them to affect our results qualitatively—they are made to simplify the model.

III. Characterization of the Equilibrium

We derive the equilibrium by first conjecturing a set of candidate stock price processes and a set of candidate trading policies and then solving for each agent's optimization problem within the candidate policy set under each candidate price process. This optimal policy is then verified to be the true optimal policy among all feasible policies. Finally, we show that the stock market clears for a particular candidate price process.[5]

A. Candidate Price Processes and Trading Policies

Without transactions costs, our model reduces to a special version of the model considered by Huang and Wang (1997) (see Sec. IV*A* for the solution to the agents' optimal trading policy and the equilibrium stock price under zero transactions costs as a special case of the model). Agents trade continuously in the stock market to hedge their nontraded risk. Because their nontraded risks always sum to zero, agents can eliminate their nontraded risk completely through trading. Therefore, the equilibrium price remains constant over time and is independent of the idiosyncratic nontraded risk as characterized by X_t. In particular, the equilibrium price has the following form:

$$P_t = p_D - p_0 \quad \forall t \geq 0, \tag{11}$$

where $p_D \equiv \bar{a}_D/r$ gives the present value of expected future dividends, discounted at the risk-free rate, and p_0 gives the discount in the stock price to adjust for risk. Because the nontraded risk is perfectly correlated with the risk of the stock, the budget constraint for an agent can be reexpressed as

$$M_t^i = \int_0^t (rM_s^i + \theta_s^i \bar{a}_D - c_s)ds + \int_0^t z_s^i \sigma_D dB_{1s} - \int \; d\theta(P + dP), \tag{12}$$

[5] Clearly, this procedure does not address the uniqueness of equilibrium, which is left for future research.

where

$$z_t^i \equiv \theta_t^i - \frac{X_t^i}{\sigma_D} \tag{13}$$

defines his net risk exposure from both his stock holding and nontraded asset. The agent's optimal trading policy is to maintain his net risk exposure at a desirable level, which is

$$z_t^i = \theta_0, \tag{14}$$

where $\theta_0 = p_0/\gamma\sigma_D^2$ is a constant. Given the form of their utility function, each agent's stock holding is independent of his wealth.[6]

In the presence of transactions costs, agents trade only infrequently. However, whenever they trade, we expect them to reach optimal risk sharing. This implies, as in the case of no transactions costs, that the equilibrium price for all trades should be the same and independent of the idiosyncratic nontraded risk X_t. Therefore, we consider the candidate stock price processes of the form (11) even in the presence of transactions costs. The discount p_0 now reflects the price adjustment of the stock for both its risk and illiquidity.

Contrary to the case of no transactions costs, it is no longer optimal to follow trading policies that always maintain the agents' net risk exposures at the desired level in the no transactions cost case (which requires continuous trading). Instead, we consider candidate trading policies that maintain each agent's net risk exposure z_t^i within a certain band. Such policies are defined by three constants, z_l, z_m, and z_u, where $z_l \le z_m \le z_u$, such that when $z_t^i \in (z_l, z_u)$, no trade occurs; when z_t^i hits the lower bound z_l, agent i buys $\delta^+ \equiv z_m - z_l$ shares of the stock and moves z_t^i to z_m; and when z_t^i hits the upper bound z_u, agent i sells $\delta^- \equiv z_u - z_m$ shares of the stock and moves z_t^i to z_m. Since $X_0 = 0$, we assume without loss of generality that $\theta_{0^-} \in (z_l, z_m)$, where θ_{0^-} is the agent's initial stock position.

Figure 1 shows an example of such a trading policy when $\sigma_X = 1$. When z_t stays within the band between $z_l = 1.6$ and $z_u = 8.4$, there is no trading and z_t follows a random walk. At the times when z_t reaches z_l or z_u, trading in the amount of $\delta^+ = 3.4$ or $\delta^- = 3.4$ occurs, respectively, and z_t is adjusted to $z_m = 5.0$, an interior point between z_l and z_u. In figure 1, over a period of 2.0 years, four trades occurred, at times $\tau_1 = 0.1$, $\tau_2 = 0.7$, $\tau_3 = 0.9$, and $\tau_4 = 1.5$ years.

We define the stopping time τ_k to be the first time the net risk exposure z_t hits the boundary of (z_l, z_m) given the agent's net risk exposure at the

[6] The agents' optimal trading policy and the equilibrium stock price under zero transactions costs are given in Sec. IV*A* as a special case of the model.

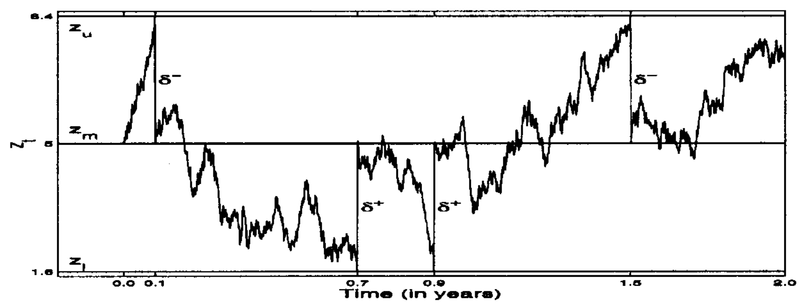

FIG. 1.—Candidate trading policy with $z_l = 1.6$, $z_u = 8.4$, $z_m = 5.0$, and $\sigma_X = 1$. The trade sizes for purchases and sales are $\delta^+ = z_m - z_l = 3.4$ and $\delta^- = z_u - z_m = 3.4$.

previous trade $z_{\tau_{k-1}} = z_m$:

$$\tau_k = \inf\{t \geq \tau_{k-1} : z_t \notin (z_l, z_u)\} \quad \forall k = 1, 2, \ldots, \tag{15}$$

where $\tau_0 = 0$. The set of stopping times $\{\tau_k : k \in \mathbb{N}_+\}$ then gives the sequence of trading times. The amount of trading at τ_k is given by

$$\delta_{\tau_k} = 1_{\{z_{\tau_k^-} = z_l\}}\delta^+ - 1_{\{z_{\tau_k^-} = z_u\}}\delta^-, \tag{16}$$

where $1_{\{\cdot\}}$ is the indicator function.

For convenience, we define a modified measure of each agent's liquid wealth,

$$\hat{M}_t^i = M_t^i + \theta_t^i p_D, \tag{17}$$

where p_D is the present value of the deterministic part of future dividends. The measure $\hat{M}_t^i$ includes agent i's bond holding plus the value of deterministic dividends from his stock holding. Thus it captures the riskless part of the agent's wealth. The risky part of the agent's wealth is determined by his net risk exposure $z_t^i = \theta_t^i - (X_t^i/\sigma_D)$. From (12), we have

$$\hat{M}_t^i = \hat{M}_0^i + \int_0^t (r\hat{M}_s^i - c_s)ds + \int_0^i z_t^i \sigma_D dB_{1s} - \sum_{\{k:\, 0 \leq \tau_k^i \leq t\}} (-p_0 \delta_k^i + \kappa_k^i). \tag{18}$$

Since $\hat{M}$ already includes the value of the stock from its deterministic dividends, trading in the stock changes $\hat{M}$ by only the marginal amount $-p_0$, the value of the stock from its uncertain dividends.

B. *The Optimal Policy*

Given the candidate stock price and trading policies, we now examine an agent's optimization problem. We start with the conjecture that each agent's value function has the form

$$J = J(\hat{M}_t, z_t, t) \equiv -e^{-\rho t - r\gamma \hat{M}_t - v(z) - \bar{v}}, \tag{19}$$

where $v(z)$ is twice-differentiable. This form of the value function is motivated by three observations. First, the agent has constant risk aversion; hence his trading policy is independent of his wealth level, which is characterized by $\hat{M}_t$. This suggests that the dependence of the value function on wealth and risk exposure may take a separable form. The conjectured value function in (19) has the form that is a product of two functions, one in $\hat{M}$ and the other in z. Second, the functional form of the utility function suggests that the value function is exponential in wealth. In particular, in the absence of any risk, we can easily verify the exponential form of the value function. Third, the agent's net risk exposure is fully characterized by z_t. Thus the dependence of his value function on his risk exposure takes the form in z.

When v is twice differentiable, the Bellman equation takes the following form:

$$0 = \sup_c \{-e^{-\rho t - \gamma c} + D[J]\}, \tag{20}$$

where $D[\cdot]$ is the standard Itô operator.[7] Within the candidate set, we now examine the optimal consumption and trading policies.

A candidate trading policy is defined by (z_l, z_m, z_u). When z_t is within the interval (z_l, z_u), the no-trade region, there is no trading and θ_t remains constant. Thus z_t simply evolves with X_t: $dz_t = -(1/\sigma_D)dX_t$. The Bellman equation reduces to

$$\rho J = \sup_c -e^{-\rho t - \gamma c} + [-r\gamma(r\hat{M} - c) + \tfrac{1}{2}(r\gamma)^2\sigma_D^2 z^2 - \tfrac{1}{2}\sigma_z^2(v'' - v'^2)]J, \tag{21}$$

where $\sigma_z^2 = \sigma_X^2/\sigma_D^2$. The first-order condition for c gives the optimal consumption policy:

$$c = \frac{1}{\gamma}[r\gamma\hat{M}_t + v(z) + \bar{v} - \ln r], \tag{22}$$

where $\bar{v} \equiv (\rho - r + r\ln r)/r$. It is trivial to verify that the second-order condition is satisfied. Substituting (22) into the Bellman equation (21)

[7] Suppose that $dx_i = a_i dt + b_i dB$, where $i = 1, 2, \ldots, m$, and $f = f(x_1, \ldots, x_m)$ is twice-differentiable. Let $f_i = \partial f/\partial x_i$, $f_{ij} = \partial^2 f/\partial x_i \partial x_j$, and $(\cdot)'$ denotes the transpose. Then

$$D[f] \equiv \sum_{i=1}^{m} a_i f_i + \tfrac{1}{2}\sum_{i,j=1}^{m} b_i(b_j)' f_{ij}.$$

Note that in this case, $dx_1 = dt$.

yields a second-order ordinary differential equation (ODE) for $v(z)$:

$$\sigma_z^2 v'' = \sigma_z^2 v'^2 + 2rv + (r\gamma)^2\sigma_D^2 z^2. \tag{23}$$

When z_t hits the boundary of the no-trade region, that is, z_l or z_u, trading occurs. The stock position, θ_t, will be adjusted to move z_t to z_m. Solving for the optimal trading policy is equivalent to finding the optimal (z_l, z_m, z_u), given the transactions costs $\kappa(\delta) = \kappa^+, \kappa^-$ ($\kappa^+ + \kappa^- = \kappa$) and price coefficient p_0.

If the trading policy (z_l, z_m, z_u) is optimal, at the trading boundaries (z_l and z_u) and with the optimal trade amounts ($\delta = \delta^+$ and δ^-, respectively), the agent must be indifferent between trading and not trading. This gives the well-known "value-matching" condition:

$$J(\hat{M}, z, t) = J(\hat{M} + p_0\delta - \kappa(\delta), z + \delta, t)$$

for $z = z_l$, z_u and $\delta = \delta^+$, δ^-, respectively, where $\delta^+ = z_m - z_l$ and $\delta^- = z_u - z_m$. Note that purchasing δ shares of the stock reduces $\hat{M}$ by $-p_0\delta$ plus the transactions cost $\kappa(\delta)$. From the conjectured form of the value function, we have

$$v(z_l) = v(z_m) - r\gamma[\kappa^+ - p_0(z_m - z_l)] \tag{24a}$$

and

$$v(z_u) = v(z_m) - r\gamma[\kappa^- + p_0(z_u - z_m)]. \tag{24b}$$

In addition, if the trading boundaries and the optimal trade amount are indeed optimal, they must satisfy the so-called "smooth-pasting" condition:

$$\frac{d}{dz}J(\hat{M}, z_l, t) = \frac{d}{dz}J(\hat{M}, z_u, t) = \frac{d}{dz}J_z(\hat{M}, z_m, t) = 0.$$

The rationale for the smooth-pasting condition is well known: if the slope of the value function at, say, z_m is not zero, we can then increase the value function by choosing a different z_m, which implies that the original z_m is not optimal. The value of $\hat{M}$ also varies with z (recall that $\hat{M}_t = M_t + \theta_t p_D = M_t + [z + (X/\sigma_D)]p_D$). In particular, a unit increase in θ reduces $\hat{M}$ by $-p_0$; hence we have $(d/dz)J = J_{\hat{M}}(r\gamma p_0) + J_z$, where $J_{\hat{M}}$ and J_z denote the partial derivatives of J with respect to $\hat{M}$ and z, respectively. From (19), we then obtain

$$v'(z_l) = v'(z_m) = v'(z_u) = -r\gamma p_0. \tag{25}$$

The value-matching condition (24) and the smooth-pasting condition (25) are two necessary conditions for a trading policy, defined by (z_l, z_m, z_u), to be optimal. They provide the boundary conditions to solve for the value function and the optimal trading policy within the candidate set.

The following theorem states that the optimal trading policy within the candidate set actually gives the optimum among all admissible policies.

THEOREM 1. Suppose that $v(z)$ satisfies (23) for $z \in [z_l, z_u]$ and (z_l, z_m, z_u) is the solution to (24) and (25), satisfying

$$z_l \leq \frac{rp_0 - \sqrt{(rp_0)^2 + \sigma_D^2 q_l}}{r\gamma\sigma_D^2}, \quad z_u \geq \frac{rp_0 + \sqrt{(rp_0)^2 + \sigma_D^2 q_u}}{r\gamma\sigma_D^2}, \tag{26}$$

where

$$q_l \equiv -(r\gamma p_0)^2\sigma_z^2 - 2r[\hat{v}(z_m) - r\gamma(\kappa^+ - p_0 z_m)]$$

and

$$q_u \equiv -(r\gamma p_0)^2\sigma_z^2 - 2r[\hat{v}(z_m) - r\gamma(\kappa^- - p_0 z_m)].$$

Then v together with (19) gives the value function for the agents' optimization problem as defined in (9) and the optimal trading policies are given by (15) and (16).

Thus solving for the agents' optimal policies reduces to solving v under the boundary conditions (24) and (25).

C. *Equilibrium Prices*

An equilibrium price process is a constant given by (11) with a particular choice of transactions cost allocation, κ^+ and $\kappa^- = \kappa - \kappa^+$, and price coefficient p_0, such that the stock market clears. Given the agents' trading policies, the market-clearing condition (10) becomes

$$\delta^+ = \delta^- \tag{27a}$$

and

$$z_m = \bar{\theta}. \tag{27b}$$

Equation (27a) implies $z_u - z_m = z_m - z_l$. The symmetry between the two agents in their exposure to nontraded risk yields $z_t^1 - z_m = z_m - z_t^2$, and their optimal trading times match perfectly when (27a) is satisfied. Furthermore, at the time of trade, the buyer wants to buy exactly the amount that the seller wants to sell. This trade amount is $\delta = \delta^+ = \delta^-$. Equation (27b) requires that both agents trade to the point at which their total holdings of the stock equal the supply. Recall that $\bar{\theta}$ is the per capita endowment of shares of the risky asset.

IV. Equilibrium Solutions

Solving for the equilibrium of the conjectured form consists of two steps. The first step is to solve for each agent's value function and optimal trading policy, given κ^+ and p_0, by solving (23) with boundary conditions (24) and (25). This is a free-boundary problem of a nonlinear ODE. The second step is to solve for κ^+ and p_0 such that the market-clearing condition (27) is satisfied. A general closed-form solution to the problem is not readily available, so we approach the problem in two ways. We first consider the special case in which transactions costs are small, for which we are able to derive approximate analytical results in subsection *C*. We then solve the general case numerically in subsection *D*. In preparation for these two approaches, we first consider the extreme cases of zero and infinite transactions costs in subsections *A* and *B*, respectively.

A. Zero Transactions Costs

When $\kappa = 0$, then $\delta^+ = \delta^- = 0$ and the agents trade continuously (as the limit of the progressively measurable trading policies given in definition 1), and we have the following result.

THEOREM 2. For $\kappa = 0$, agent i's optimal trading policy under a constant stock price $P_t = (\bar{a}_D/r) - p_0$ is $\theta_t^i = \bar{z}_m + (X_t^i/\sigma_D)$, where $\bar{z}_m = p_0/(\gamma\sigma_D^2)$, and his value function is

$$J_t = -\exp\left[-\rho t - r\gamma(\hat{M}_t^i - p_0\bar{z}_m) - \tfrac{1}{2}r\gamma^2\sigma_D^2\bar{z}_m^2(1-\gamma^2\sigma_X^2) - \bar{v}\right]. \quad (28)$$

Moreover, in equilibrium, $p_0 = \bar{p}_0 \equiv \gamma\sigma_D^2\bar{\theta}$ and $\bar{z}_m = \bar{\theta}$.

Theorem 2 has two parts: the first part gives the agents' optimal trading policy, including the average demand for the stock for a given price level $(\bar{a}_D/r) - p_0$ or p_0, and the second part gives the equilibrium stock price p_0 that clears the market, that is, $z_m = \bar{\theta}$.

In particular, agent i's stock holding has two components. The first component, $\bar{z}_m$, which is constant, gives his unconditional stock position. For $P_t = (\bar{a}_D/r) - p_0$, the expected excess return on one share of stock is rp_0 and the return variance is σ_D^2. Hence, rp_0/σ_D^2 gives the price per unit risk of the stock. Moreover, agent i's risk aversion (toward uncertainty in his wealth) is $r\gamma$. Thus his unconditional stock position, $\bar{z}_m = (rp_0/\sigma_D^2)/r\gamma = p_0/(\gamma\sigma_D^2)$, is proportional to his risk tolerance and the price of risk. The second component of agent i's stock position is proportional to X_t^i, his exposure to the nontraded risk. This component reflects his hedging position against nontraded risk, and the proportionality coefficient, $1/\sigma_D$, gives the hedge ratio.

In equilibrium, market clearing requires that $\bar{z}_m = \bar{\theta}$; hence $p_0 =$

$\bar{p}_0 \equiv \gamma\sigma_D^2\bar{\theta}$. As mentioned earlier, p_0 gives the discount in the price of the stock for its risk and illiquidity. In the absence of transactions costs, the stock is liquid and $p_0 = \bar{p}_0$. Thus $\bar{p}_0$ can be interpreted as the risk discount of the stock. In the presence of transactions costs, we define the difference between p_0 and $\bar{p}_0$, denoted by π,

$$\pi \equiv p_0 - \bar{p}_0, \tag{29}$$

to be the *illiquidity discount* of the stock.

B. *Infinite Transactions Costs*

To develop an intuition about the illiquidity discount and to put a bound on its magnitude, we first consider the extreme case in which the transactions costs are prohibitively high except at $t = 0$. That is, $\kappa = \hat{\kappa}1_{\{t>0\}}$, where $\hat{\kappa} \to \infty$. Agents can trade at zero cost at $t = 0$ but cannot trade afterward.[8] This case can be solved in closed form, and we have the following result.

THEOREM 3. For $\kappa = \hat{\kappa}1_{\{t>0\}}$, where $\hat{\kappa} \to \infty$, agent i's stock demand is

$$\theta_0^i = \tfrac{1}{2}(1 + \sqrt{1 - 4\gamma^2\sigma_X^2})\frac{p_0}{\gamma\sigma_D^2} + \frac{X_0^i}{\sigma_D}.$$

In equilibrium, $\theta_t^i = \bar{\theta}$, and the stock price at $t = 0$ is $P_0 = (\bar{a}_D/r) - p_0$, where

$$p_0 = \bar{p}_0\left[1 + \frac{4\gamma^2\sigma_X^2}{(1 + \sqrt{1 - 4\gamma^2\sigma_X^2})^2}\right]$$

and $\bar{p}_0$ is given in theorem 2.

In this case, the stock becomes completely illiquid after the initial trade. At the same price, the demand for stock is lower than in the case in which $\kappa = 0$. In equilibrium, an illiquidity discount in its price is required:

$$\hat{\pi} \equiv \bar{p}_0\frac{4\gamma^2\sigma_X^2}{(1 + \sqrt{1 - 4\gamma^2\sigma_X^2})^2},$$

and for σ_X^2 small, we have $\hat{\pi} \approx \gamma^2\sigma_X^2\bar{p}_0$.

This extreme case illustrates three points. First, the agents' inability to trade in the future reduces their current demand for the stock. As a result, its price carries an additional discount in equilibrium to compensate for illiquidity (see also Hong and Wang 2000). Second, this

[8] This situation has been considered by Hong and Wang (2000) when they analyze the effect of market closures on asset prices, which is equivalent to imposing prohibitive transactions costs.

illiquidity discount is proportional to agents' high-frequency trading needs, which is characterized by the instantaneous volatility of their nontraded risk, σ_X^2. Third, the liquidity discount also increases with the risk of the stock, which is measured by σ_D^2.

C. *Small Transactions Costs: An Approximate Solution*

When the transactions costs are finite, agents can trade after the initial date (at a cost) and the stock becomes more liquid. We expect the magnitude of the illiquidity discount to be smaller than in the extreme case above. However, the qualitative nature of the results remains the same, as we show below.

For tractability, we consider the case in which the transactions costs are small. We seek the solution to each agent's value function, optimal trading policy, the equilibrium cost allocation, and stock price that can be approximated by powers of $\epsilon \equiv \kappa^{\nu}$, where ν is a positive constant. In particular, v takes the form $v(z, \epsilon)$ and $\kappa^{\pm}$ takes the form

$$\kappa^{\pm} = \kappa\left[\tfrac{1}{2} \pm \sum_{n=1}^{\infty} k^{(n)}\epsilon^{n}\right], \tag{30}$$

where $k^{(n)}$ are constants to be determined. We also use $o(\kappa^{\nu})$ to denote terms of higher order than κ^{ν} and $O(\kappa^{\nu})$ to denote terms of the same order as κ^{ν}. The following theorem summarizes our results on optimal trading policies.

THEOREM 4. Let $\epsilon \equiv \kappa^{1/4}$. For (*a*) κ small and $\kappa^{\pm}$ in the form of (30) and (*b*) $v(z, \epsilon)$ analytic for small z and ϵ, an agent's optimal trading policy is given by

$$\delta^{\pm} = \phi\kappa^{1/4} \pm \frac{6}{11}\left[k^{(1)} - \frac{2}{15}r\gamma p_0\phi\right]\phi\kappa^{1/2} + o(\kappa^{1/2}) \tag{31a}$$

and

$$z_m = \frac{p_0}{\gamma\sigma_D^2} + \frac{4}{11}\left[k^{(1)} - \frac{71}{120}r\gamma p_0\phi\right]\phi\kappa^{1/2} + o(\kappa^{1/2}), \tag{31b}$$

where $\phi = [6/(r\gamma)]^{1/4}(\sigma_X/\sigma_D^2)^{1/2}$.

Here, δ^{+} and δ^{-} are the same to the first order of $\epsilon = \kappa^{1/4}$ but differ in higher orders of ϵ.

The stock market equilibrium is obtained by choosing $\kappa^{\pm}$, that is, $k^{(1)}$, …, and p_0 such that the market-clearing condition (27) is satisfied, yielding the following theorem.

THEOREM 5. For (*a*) κ small and $\kappa^{\pm}$ in the form of (30), (*b*) $v(z, \epsilon)$ analytic for small z and ϵ, and (*c*) $p(\epsilon)$ analytic for small ϵ, the equilibrium stock price and transactions cost allocation are given by

$$p_0 = \gamma\sigma_D^2\bar{\theta}(1 + \tfrac{1}{6}r\gamma^2\sigma_D^2\phi^2\kappa^{1/2}) + o(\kappa^{1/2}) \tag{32a}$$

and

$$\kappa^{\pm} = \kappa\left[\frac{1}{2} \pm \frac{2}{15}r\gamma p_0\phi\kappa^{1/4} + o(\kappa^{1/4})\right], \tag{32b}$$

and the equilibrium trading policies are given by (31), with the equilibrium value of p_0 and $\kappa^{\pm}$.

D. *General Transactions Costs: A Numerical Solution*

In the general case in which κ can take arbitrary values, we have to solve both the optimal trading policy and the equilibrium stock price numerically. Given p_0 and $\kappa^{\pm}$, we can solve (23), (24), and (25) for each agent's optimal trading policy. We can then solve for values of p_0 and $\kappa^{\pm}$ that satisfy the market-clearing condition (10).

In the examples throughout this paper, we use parameter values obtained from a calibration exercise described in Section VI. In particular, we set $\rho = 0.1$, $\gamma = 1.5$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = 1.0$, and $\bar{\theta} = 5.0$.

Figure 2 shows the numerical solution for the optimal trade size as a function of transactions costs, where we have chosen $\kappa^{\pm}$ such that $\delta^+ = \delta^- \equiv \delta$. This choice of $\kappa^{\pm}$ is arbitrary—merely to illustrate the agents' trading policy—and does not necessarily correspond to any market equilibrium. In figure 2*a*, δ is plotted against the value of κ. Each circle represents the value of δ for a particular value of κ. In figure 2*b*, δ is plotted against the value of $\kappa^{1/4}$. This transformation is motivated by the approximate solution when κ is small. For comparison, we have also plotted the analytical approximation obtained for small κ (the solid lines).

Figure 3 shows the numerical solution for $v(z)$ for $p_0 = 0.6105$ and $\kappa^+ = \kappa^- = \kappa/2 = 0.0039$, which defines the value function. For convenience, we have plotted $\tilde{v}(z) \equiv v(z) - r\gamma p_0(z - z_m)$ instead of $v(z)$ itself. By the boundary conditions for $v(z)$, $\tilde{v}(z)$ must have zero slope at z_l, z_m, and z_u and the same value at z_l and z_u (see also eq. [33]).

Given the solution to the agents' optimal trading policies, we can search for the p_0 and $\kappa^{\pm}$ such that the market-clearing condition (27) is satisfied. Figure 4 plots the numerical solution (circles) and the analytical approximation (solid line) for the illiquidity discount π in the stock price (recall that $\pi = p_0 - \bar{p}_0$) for various values of the transactions cost. In figure 4*a*, π is plotted against κ. In figure 4*b*, π is plotted against

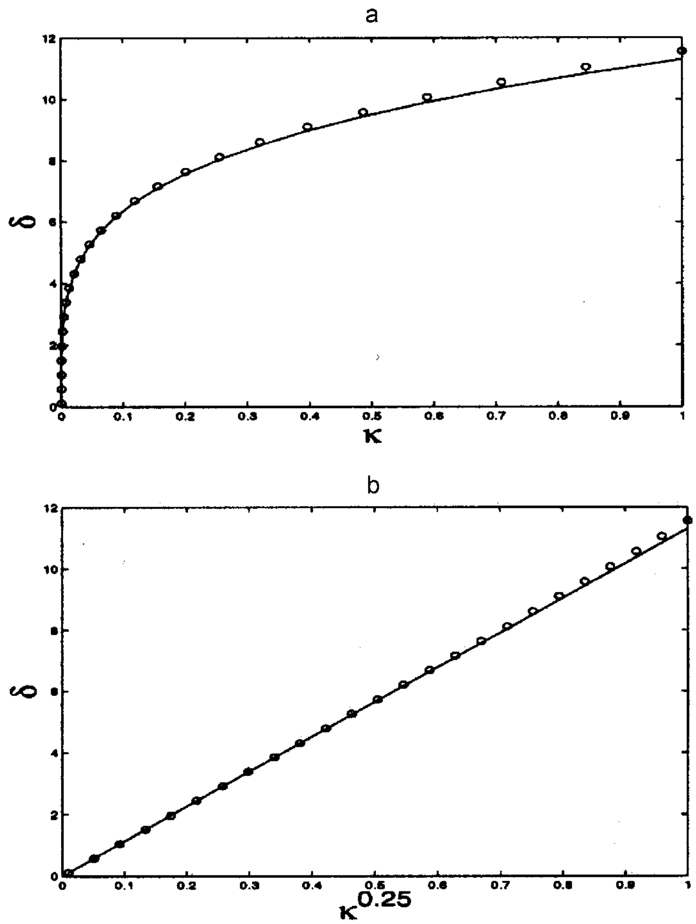

FIG. 2.—Trade size δ plotted against transactions costs κ (a) and $\kappa^{1/4}$ (b). The circles represent the numerical solution, and the solid line plots the analytical approximation. The parameter values are $\rho = 0.1$, $\gamma = 1.5$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = 1.0$, and $\bar{\theta} = 5.0$. Given the parameter values, $\phi = 11.61$.

$\kappa^{1/2}$. It is interesting to note that the analytic approximation obtained for small values of transactions costs still fits quite well even for fairly large values of κ.

V. Analysis of Equilibrium

We now discuss in more detail the impact of transactions costs on agents' trading policies, the equilibrium stock price, and trading volume. We

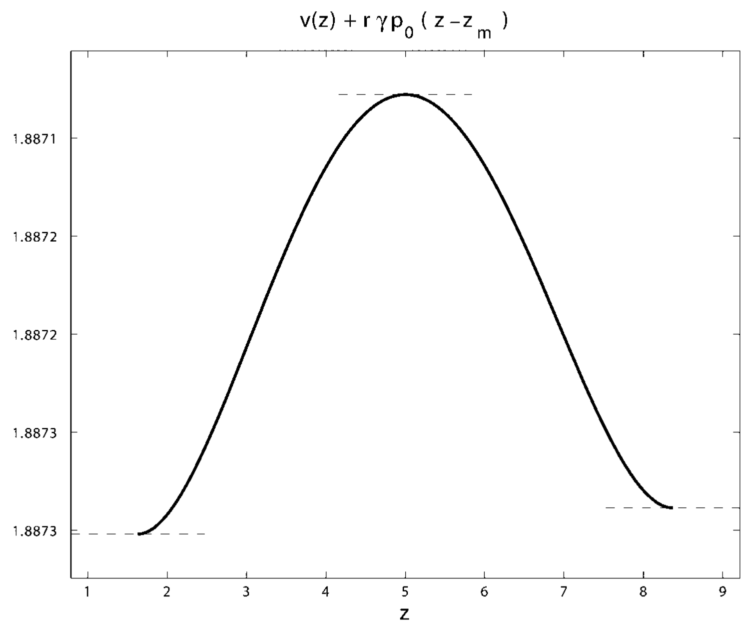

FIG. 3.—Function $v(z) + r\gamma p_0(z - z_m)$. Here we have set $p_0 = \gamma\sigma_D^2\bar{\theta} = 0.6105$ and $\kappa^+ = \kappa^- = \kappa/2 = 0.0039$. The other parameter values are $\rho = 0.1$, $\gamma = 1.5$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = 1.0$, and $\bar{\theta} = 5.0$.

focus on the case in which κ is small. For convenience, we maintain terms only up to the lowest appropriate order of κ in our discussion.

A. *Trading Policy*

When transactions costs are zero ($\kappa = 0$), agent i trades continuously in the stock as his exposure to nontraded risk changes to maintain his net risk exposure at the optimal level of $z_t^i = \theta_t^i - (X_t^i/\sigma_D) = \bar{z}_m = p_0/(\gamma\sigma_D^2)$ (see theorem 2). When transactions costs are positive, it becomes costly to maintain $z_t^i = \bar{z}_m$ at all times. Instead, he does not trade when z_t^i is not too far away from a desirable position z_m, which defines a no-trade region $(z_l, z_u) = (z_m - \delta^+, z_m + \delta^-)$. However, when z_t^i hits the boundary of the no-trade region, agent i trades the required amount (δ^+ or δ^-) to bring z_t^i back to the optimal level z_m. Two sets of parameters characterize the agent's optimal trading policy: the widths of the no-trade region, δ^+ and δ^-, and the base level to which he trades, z_m, when he does trade. In general, z_m is different from $\bar{z}_m$, the position to which he would trade in the absence of transactions costs. We now discuss these two sets of parameters separately.

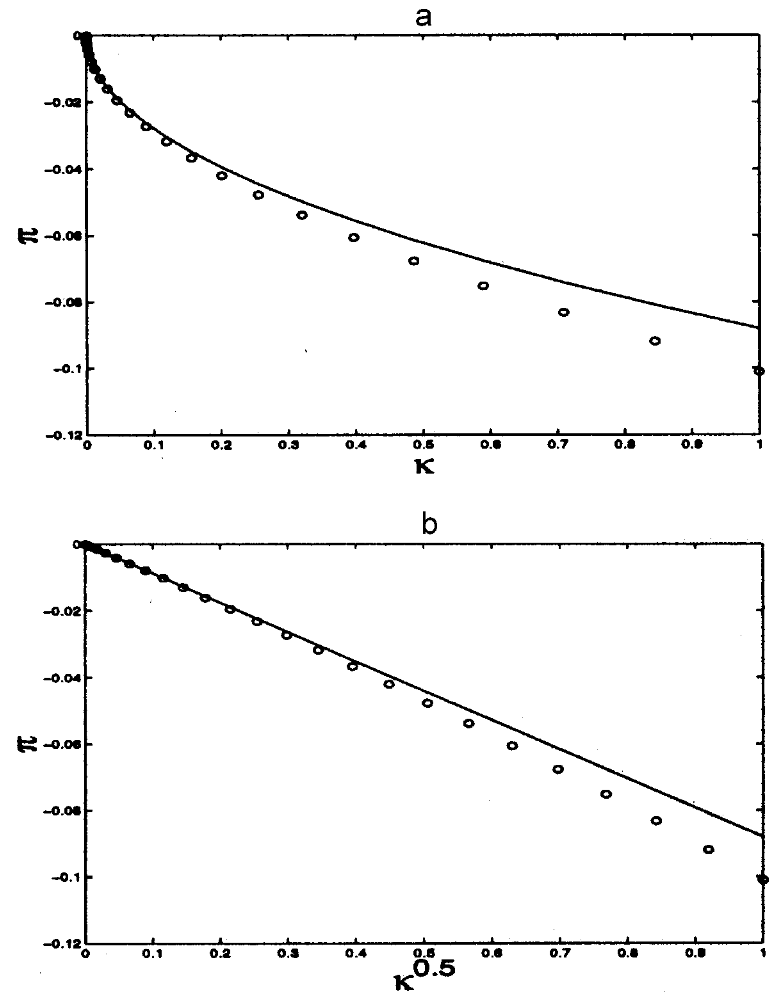

FIG. 4.—Illiquidity discount π plotted against κ (a) and $\kappa^{1/2}$ (b). The circles represent the numerical solution. The solid line plots the analytical approximation. The parameter values are $\rho = 0.1$, $\gamma = 1.5$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = 1.0$, and $\bar{\theta} = 5.0$.

To the lowest order of κ, $\delta^+ = \delta^- = \phi\kappa^{1/4}$ as shown in theorem 4. In other words, the width of the no-trade region exhibits a "quartic-root law" for small transactions costs, which arises from the boundary conditions. To see why, observe that to the lowest order of κ, $\kappa^+ = \kappa^- = \kappa/2$. Using $\tilde{v}(z) \equiv v(z) + r\gamma p_0(z - z_m)$, we can reexpress the boundary conditions in (24) and (25) in $\tilde{v}$:

$$\tilde{v}(z_m - \delta^+) - \tilde{v}(z_m) = -\frac{r\gamma\kappa}{2} = \tilde{v}(z_m + \delta^-) - \tilde{v}(z_m) \tag{33a}$$

and

$$\tilde{v}'(z_m - \delta^+) = \tilde{v}'(z_m) = \tilde{v}'(z_m + \delta^-) = 0. \tag{33b}$$

The symmetry between the boundary conditions for the upper and lower no-trade band immediately implies that $\delta^+ = \delta^- = \delta$. Moreover, $\tilde{v}$ is symmetric around z_m. A Taylor expansion of the two boundary conditions around z_m yields (to the lowest order of κ)

$$\frac{1}{2!}\tilde{v}_2\delta^2 + \frac{1}{4!}\tilde{v}_4\delta^4 = -\frac{r\gamma\kappa}{2} \tag{34a}$$

and

$$\tilde{v}_2\delta + \frac{1}{3!}\tilde{v}_4\delta^3 = 0, \tag{34b}$$

where $\tilde{v}_\kappa$ denotes the kth derivative of $\tilde{v}$ at z_m. Then $\tilde{v}_2 = -(1/3!)\tilde{v}_4\delta^2$ follows immediately from equation (34b) and $\delta = (12r\gamma/\tilde{v}_4)^{1/4}\kappa^{1/4} \propto \kappa^{1/4}$ follows immediately from equation (34a), suggesting that the quartic-root relation between δ and κ for small κ is determined by the boundary conditions. As demonstrated in Section III, the boundary conditions are merely optimality conditions under the form of the transactions cost. For this reason, the quartic-root relation between the width of the no-trade region and the fixed transactions cost may be a more general property of optimal trading policies under fixed transactions costs.[9]

Having established that the width of the no-trade region should be proportional to the quartic root of κ (i.e., $\delta = \phi\kappa^{1/4}$), we now examine the proportionality coefficient ϕ. From theorem 4, we have $\phi = [6\sigma_z^2/(r\gamma\sigma_D^2)]^{1/4}$. Note that $r\gamma\sigma_D^2$ corresponds to the certainty equivalence of the (per unit of time) expected utility loss for bearing the risk of one stock share. It is then not surprising that ϕ (and δ) is negatively related to $r\gamma\sigma_D^2$. Moreover, σ_z^2 gives the variability of the agent's nontraded risk. For larger σ_z^2, the agent's hedging need is changing more quickly. Given the cost of changing his hedging position, the agent is

[9] The above results on optimal trading policies under fixed transactions costs are closely related to the results of Atkinson and Wilmott (1995) and Morton and Pliska (1995). Morton and Pliska solve for the optimal trading policy when an agent maximizes his asymptotic growth rate of wealth and pays a cost as a fixed fraction of his total wealth for each transaction. Atkinson and Wilmott show that when the transactions cost, as a fraction of the total wealth, is small, the no-trade region is proportional in size to the fourth root of the transactions cost.

more cautious in trading on immediate changes in his hedging need. Thus ϕ (and δ) is positively related to σ_z^2.

Under the optimal trading policy, agents trade only infrequently. Define $\Delta\tau \equiv \mathrm{E}[\tau_{k+1} - \tau_k]$ to be the average time between two neighboring trades. It is easy to show that

$$\Delta\tau = \frac{\delta^2}{\sigma_z^2} \approx \left(\frac{\phi^2}{\sigma_z^2}\right)\kappa^{1/2} = \left(\frac{6}{r\gamma\sigma_X^2}\right)^{1/2}\kappa^{1/2} \tag{35}$$

(see, e.g., Harrison 1990). Not surprisingly, the average waiting time between trades is inversely related to σ_X, the volatility of the agent's nontraded risk, his risk aversion γ, and the risk he has to bear between trades, σ_D. Moreover, it is proportional to the square root of the transactions cost. Figure 5 plots the average trading interval $\Delta\tau$ versus different values of transactions cost κ as well as the appropriate power law for small κ's.

The power laws derived above for the impact of transactions cost on trade size and trading frequency, $\delta \propto \kappa^{1/4}$ and $\Delta\tau \propto \kappa^{1/2}$, imply a power law between trade size and trading frequency:

$$\delta \propto (\Delta\tau)^{1/2}, \tag{36}$$

which has been empirically tested and confirmed by Lo, Mamaysky, and Wang (2003).

When each agent chooses to trade, he trades to a base position z_m. In the absence of transactions costs, each agent trades to a position $\bar{z}_m$ that is most desirable given his current nontraded risk. As his nontraded risk changes, he maintains this desirable position by constantly trading. In the presence of transactions costs, however, an agent trades only infrequently. A position that is desirable now becomes less desirable later. But he has to stay in this position until the next trade. Anticipating this state of affairs, the agent chooses a position now that takes into account the inability to revise it easily later.

From theorems 4 and 5, the shift in the base position is given by $\Delta z_m \equiv \bar{z}_m - z_m = \frac{1}{6}r\gamma p_0(\sigma_z^2\Delta\tau)$. It is not surprising that Δz_m is proportional to the total volatility of an agent's nontraded risk over the no-trade period, which is $\sigma_z^2\Delta\tau$. Moreover, Δz_m is proportional to p_0, the risk discount on the stock. To develop additional intuition for this result, consider the following heuristic argument. Suppose that the current level of the agent's nontraded asset is zero. The uncertainty in its level over the next no-trade period, denoted by $\tilde{z}$, gives rise to an additional uncertainty in his wealth, $-\tilde{z}(-p_0 + \tilde{d})$, where $\tilde{d}$ denotes the stock dividend over the period (we set $\sigma_D = 1$ for simplicity). Although $\tilde{z}$ has zero mean, its impact on the overall uncertainty in wealth is not zero. When we average over $\tilde{z}$—assumed to be normally distributed with var-

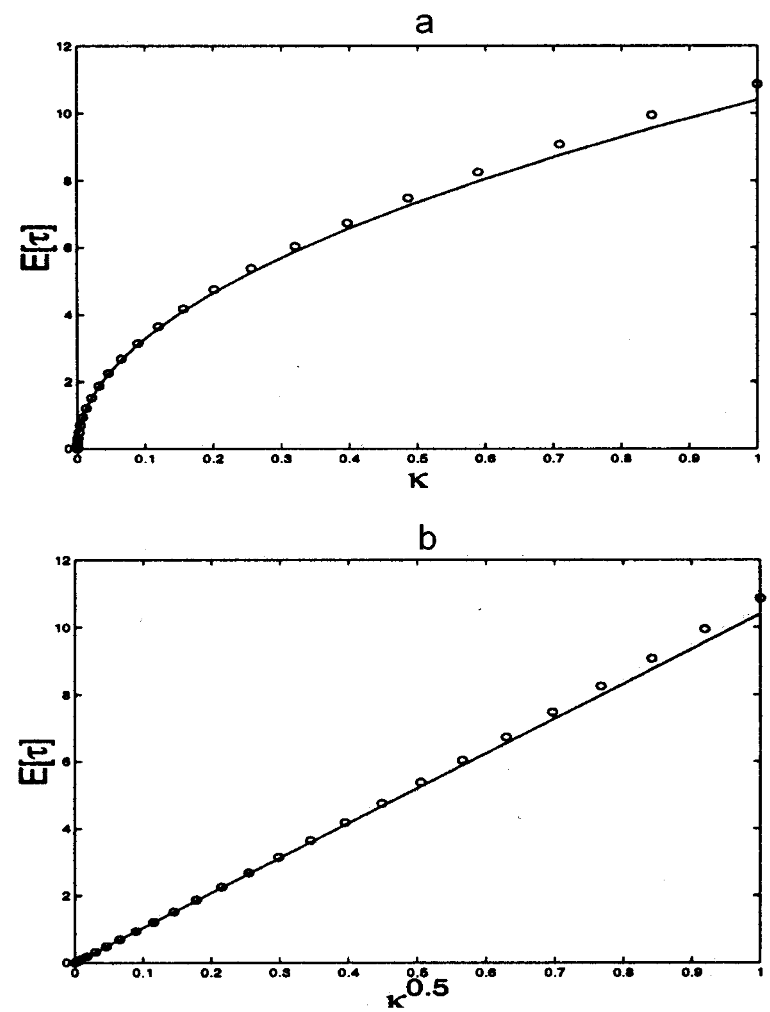

FIG. 5.—Trading interval. The two panels show the expected interarrival times plotted against κ (a) and its square root (b), respectively. The circles represent the numerical solution. The solid line plots the analytical approximation. The parameter values are $\rho = 0.1$, $\gamma = 1.5$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = 1.0$, and $\bar{\theta} = 5.0$.

iance σ_z^2—the agent's utility over his future wealth is proportional to

$$\mathrm{E}_{\tilde{z}}[-e^{-r\gamma(\theta-\tilde{z})(-p_0+\tilde{d})}] = -e^{-r\gamma[\theta-(1/2)r\gamma(-p_0+\tilde{d})\sigma_z^2\Delta\tau](-p_0+\tilde{d})},$$

where $\mathrm{E}_{\tilde{z}}$ denotes the expectation with respect to $\tilde{z}$, θ is the agent's stock position, and $\Delta\tau$ is the length of the no-trade period. In other words, the uncertainty in $\tilde{z}$ leads to an effective risk exposure in the agent's wealth that is equivalent to an average stock position of size $\frac{1}{2}r\gamma p_0(\sigma_z^2\Delta\tau)$, which is proportional to p_0 because the uncertainty in wealth generated by uncertainty in $\tilde{z}$ is proportional to p_0. Consequently, the agent reduces his base stock position by the same amount. This shift in the agent's base position reflects the decrease in his demand for the stock in response to its illiquidity.

B. Stock Prices and the Illiquidity Discount

In equilibrium, the stock price has to adjust in response to the negative effect of illiquidity on agents' stock demand, giving rise to an illiquidity discount π. For small transactions costs, the illiquidity discount is proportional to the square root of κ. Figure 4 shows that this square root relation provides a reasonable approximation even for fairly large transactions costs. From theorem 5, we have

$$\pi \approx \gamma\sigma_D^2\Delta z_m \approx \frac{1}{6}\gamma(r\gamma\sigma_D^2)\bar{p}_0(\sigma_z^2\Delta\tau) \approx \frac{1}{\sqrt{6}}r^{1/2}\gamma^{3/2}\sigma_X\bar{p}_0\kappa^{1/2}. \qquad (37)$$

As we have shown, fluctuations in his nontraded risk and the cost of adjusting stock positions to hedge this risk reduce an agent's stock demand by Δz_m. Given the linear relation between the agents' stock demand and the stock price, the price has to decrease proportionally to the decrease in demand to clear the market, which gives the illiquidity discount in the first expression of (37). Moreover, the decrease in agents' stock demand is proportional to the total risk discount of the stock (p_0) and the volatility of their nontraded risk between trades ($\sigma_z^2\Delta\tau$), which leads to the second expression in (37).

The last expression in (37) expresses the illiquidity discount of the stock in terms of the underlying parameters of the model. The illiquidity discount increases with the exposure to nontraded risk and its volatility σ_X. Moreover, it is proportional to the cubic power of γ. Compared to risk discount, which is proportional to γ, we infer that the illiquidity discount is highly sensitive to the agents' risk aversion.

Using a model similar to ours but with *proportional* transactions costs and deterministic trading needs, Vayanos (1998) finds that the illiquidity discount in the stock price is linear in the transactions costs when they are small. Our result shows that small fixed transaction costs can give rise to a nontrivial illiquidity discount when agents have high-frequency trading needs. Given the difference in the nature of transactions costs between our model and Vayanos's, our result is not directly comparable to his. However, our result does suggest that the presence of high-frequency trading needs is important in analyzing the effect of transactions costs on asset prices. We return to this point in subsection *D*.

C. *Trading Volume*

Economic intuition suggests that an increase in transactions costs must reduce the volume of trade. Our model suggests a specific form for this relation. In particular, the equilibrium trade size is a constant. From our solution to equilibrium, the volume of trade between time interval t and $t+1$ is given by

$$\nu_{t+1} = \sum_{\{k:\, t<\tau_k\leq t+1\}} |\delta_k^i|, \tag{38}$$

where $i = 1$ or 2. The average trading volume per unit of time is

$$\mathrm{E}[\nu_{t+1}] = \mathrm{E}\left[\sum_k 1_{\{\tau_k\in(t,t+1]\}}\right]\delta \equiv \omega\delta,$$

where ω is the frequency of trade, that is, the number of trades per unit of time. For convenience, we define another measure of average trading volume: the number of shares traded per average trading time, or

$$\nu = \frac{\delta}{\Delta\tau} = \frac{\sigma_z^2}{\delta}, \tag{39}$$

where $\Delta\tau \equiv \mathrm{E}[\tau_{k+1} - \tau_k] \approx \delta^2/\sigma_z^2$ is the average time between trades. From (31), we have

$$v = \sigma_z^2\phi^{-1}\kappa^{-1/4}[1 + O(\kappa^{1/4})] = \left(\frac{r\gamma}{6}\right)^{1/4}(\sigma_X)^{3/2}\sigma_D^{-1}\kappa^{-1/4}[1 + O(\kappa^{1/4})],$$

which increases with risk aversion. Clearly, as κ goes to zero, trading volume goes to infinity. However, we also have

$$\frac{\Delta\nu}{\nu} \approx -\frac{1}{4}\frac{\Delta\kappa}{\kappa}.$$

In other words, for positive transactions costs, a 1 percent increase in the transactions cost decreases trading volume by only 0.25 percent. In this sense, for positive transactions costs, an increase in the cost reduces the volume only mildly at the margin. Figure 6 plots the average-volume measure ν versus different values of transactions cost κ as well as the appropriate power laws.

D. *High-Frequency versus Low-Frequency Trading Needs*

In our previous discussion, we have emphasized that the presence of high-frequency trading needs significantly enhances the impact of transactions costs on asset prices. The intuition is straightforward: if agents need to trade frequently for hedging or portfolio-rebalancing purposes,

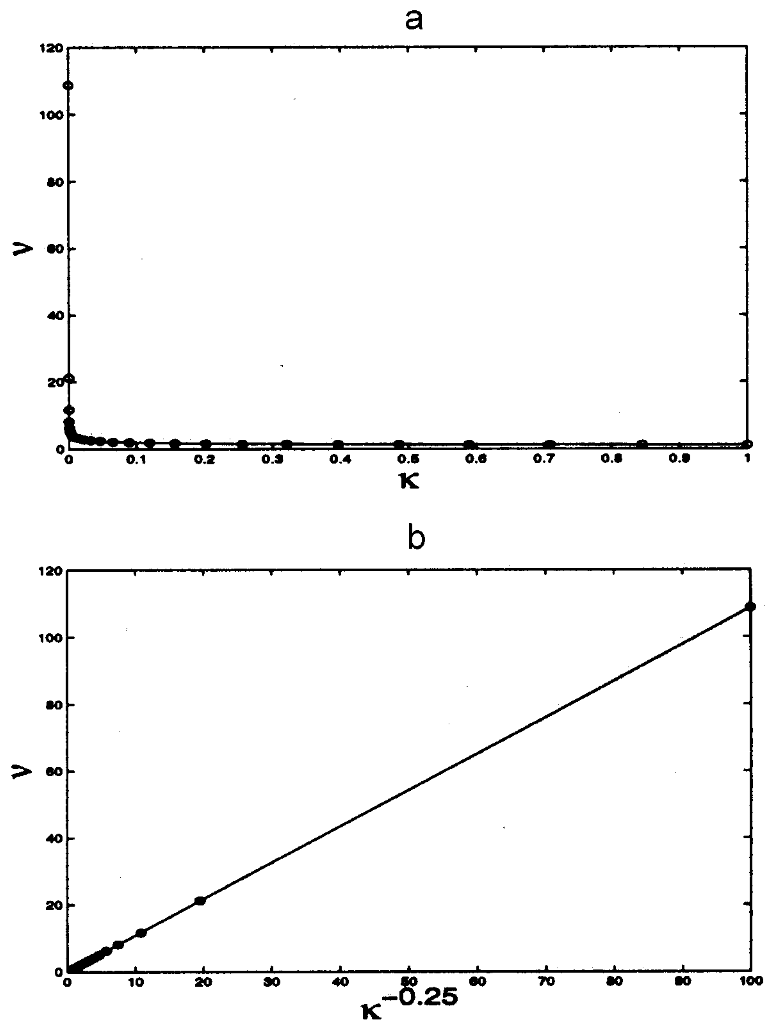

FIG. 6.—Trading volume. The two panels show the volume measure ν plotted against κ (a) and $\kappa^{-1/4}$ (b). The parameter values are $\rho = 0.1$, $\gamma = 1.5$, $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, $\sigma_X = 1.0$, and $\bar{\theta} = 5.0$.

then small transactions costs will have large effects on asset prices. If, on the other hand, agents do not need to trade frequently, then transactions costs will have little impact on asset prices.

To confirm this intuition explicitly, we consider a variation of the model in Section II in which agents have only low-frequency trading needs and examine how transactions costs affect prices in that case.

Specifically, let

$$X_t = \bar{a}_X t, \tag{40}$$

where $\bar{a}_X$ is a positive constant. Thus agent 1's exposure to nontraded risk increases deterministically over time at a constant rate of $\bar{a}_X$: $-X_t^1 = X_t = \bar{a}_X t$, and agent 2's exposure decreases over time at the same rate. As in the case in which X_t is stochastic, each agent will trade in the stock to maintain his net risk exposure $z_t^i = \theta_t^i - (X_t^i/\sigma_D)$ at a certain constant level.

In the absence of transactions cost, both agents trade continuously but deterministically. In particular, agent 1 will sell shares of the stock at a deterministic rate of $\bar{a}_X$ so that his net exposure is $z_t^1 = \bar{z}_m = p_0/\gamma\sigma_D^2$, as in Section IVA. In this case, trading volume is finite, even without transactions costs. This is in sharp contrast to the case in which X_t is stochastic, where trading volume is unbounded if transactions costs are zero. For deterministic X_t, there are no high-frequency trading needs. Now in the presence of transactions costs, agents trade only infrequently. In particular, agent i does not trade as long as z_t^i remains within a no-trade region. He trades only when z_t^i reaches the boundary of the no-trade region to bring it back to the optimal level. For example, agent 1's net risk exposure z_t^1 increases at a constant rate of $\bar{a}_X$. His optimal trading policy is defined by $[z_m^1, z_u^1]$. When z_t^1 is in $[z_m^1, z_u^1)$, he does not trade. However, when z_t^1 hits z_u^1, he sells $\delta^{1-} \equiv z_u^1 - z_m^1$ shares of the stock and z_t^1 is moved back to z_m. The same process is then repeated over time. The optimal policy of agent 2, which is defined by $[z_l^2, z_m^2]$, is just the opposite, with infrequent but repeated share purchases of $\delta^{2+} \equiv z_m^2 - z_l^2$ when z_t^2 hits z_l^2. It is obvious that the trading policies here are qualitatively the same as those when z_t^i is stochastic. The only difference is that since z_t^i drifts deterministically in one direction, the no-trade region and the trades are one-sided.

Using the same approach as before, we can solve for the equilibrium in the case in which there are no high-frequency trading needs, and the results are summarized in the following theorem.

Theorem 6. Let $\epsilon = \kappa^{1/3}$. For (*a*) $X_t = \bar{a}_X t$ ($\bar{a}_X \geq 0$), (*b*) $\kappa^{\pm} = \kappa/2$, (*c*) $v(z, \epsilon)$ analytic for small z and ϵ, and (*d*) $p(\epsilon)$ analytic for small ϵ, the agents' optimal trading policies are given by

$$\delta^{1+} = \lambda\kappa^{1/3} + o(\kappa^{2/3}), \quad \delta^{2-} = \delta^{1+}, \tag{41a}$$

and

$$z_m^1 = \frac{p_0}{\gamma\sigma_D^2} + \tfrac{1}{2}\lambda\kappa^{1/3} - \tfrac{1}{2}(\lambda\gamma\sigma_D^2)^{-1}\kappa^{2/3} + o(\kappa^{2/3}),$$

$$z_m^2 = \bar{\theta} - (z_m^1 - \bar{\theta}), \tag{41b}$$

where $\lambda = [6/(r\gamma\sigma_D^3)]^{1/3}(\bar{a}_X)^{1/3}$. The equilibrium stock price is given by $p_0 = \bar{p}_0 + o(\kappa)$.

From (41a), it is clear that the no-trade region is proportional to $\kappa^{1/3}$. This is smaller than the size of the no-trade region in the presence of high-frequency trading needs, which is proportional to $\kappa^{1/4}$. The intuition behind this result is straightforward: high-frequency trading needs are generated by stochastic variations in the agents' risk exposure. The stochastic nature of their trading needs deters them from trading too quickly in response to their instantaneous risk exposure. After all, there is a significant chance that an agent's exposure moves in the opposite direction in the next instant. As a result, he allows a large no-trade region. For low-frequency trading needs caused by deterministic shifts X_t, future trades are more predictable; hence each agent is willing to trade more promptly as his risk exposure changes, leaving a smaller no-trade region. A smaller no-trade region implies that the agent bears less risk between trades. Consequently, the decrease in his stock demand due to no trading is smaller than in the case of high-frequency trading needs. In equilibrium, the illiquidity discount is also smaller. In fact, the illiquidity discount is negligible to the first order of κ. In other words, the impact of the transactions cost on the stock price is very small when κ is small.

The comparison between the two cases, one with high-frequency trading needs and one without, confirms the intuition that the impact of transactions costs on asset prices becomes more significant when there is need to trade more frequently.

VI. A Calibration Exercise

Our model shows that even small fixed transactions costs imply a significant reduction in trading volume and an illiquidity discount in asset prices. To develop additional insights into the practical relevance of fixed costs for asset markets, we calibrate our model using empirically plausible parameter values and derive numerical implications for the illiquidity discount, trading frequency, and trading volume. From (37), for small fixed costs κ, the illiquidity premium π is

$$\pi = \left(\frac{1}{\sqrt{6}}\right) r^{1/2}\gamma^{3/2}\sigma_X\bar{p}_0\kappa^{1/2}.$$

The parameters to be calibrated include the interest rate r, the risk discount $\bar{p}_0$, the agents' coefficient of absolute risk aversion γ, the volatility of the nontraded risk σ_X, and the fixed transactions cost κ.

In our model, dividends and stock returns follow Gaussian processes. In particular, the annual dividend is independently normally distributed with a mean of $\bar{a}_D$ and a volatility of σ_D. The annual stock return is also normally distributed with a mean of $a_D/\bar{P}$ (where $\bar{P} \equiv [\bar{a}_D/r] - \bar{p}_0$ is the price level in the absence of transactions costs) and a volatility that is the same as that of the dividend. On the basis of the annual time series of U.S. real stock prices and dividends from 1871 to 1986, Campbell and Kyle (1993) estimated a detrended Gaussian model for the dividend and return on the aggregate stock market. Using their estimates, we can calibrate the real interest rate r, the average dividend rate $\bar{a}_D$, the dividend volatility σ_D, and the price level $\bar{P}$ in our model.[10] In particular, we use the following parameter values: $r = 0.0370$, $\bar{a}_D = 0.0500$, $\sigma_D = 0.2853$, and $\bar{P} = 0.7409$. These parameter values correspond to an average annual dividend yield of $\bar{a}_D/\bar{P} = 0.0675$ and a volatility of $\sigma_D/\bar{P} = 0.3851$ in our model.

The remaining parameters to be specified are γ, σ_X, and κ. There is little empirical consensus on their values, so we consider a range of values for each: $\gamma = 0.5$, 1.0, and 2.5; $\sigma_X = 0.2$, 1.0, and 5.0; and $\kappa/\bar{P} = 0.1$ percent, 0.5 percent, 1.0 percent, and 5.0 percent, where we have expressed the transactions cost as a fraction of the share price of the stock $\bar{P}$ to make it somewhat easier to interpret. Since κ is a fixed cost, its value is, by definition, scale-dependent and must therefore be considered in the context of the calibration exercise.

Table 1 summarizes the results of our calibrations in five panels, each containing different variables of interest for different values of γ, σ_X, and $\kappa/\bar{P}$. Panel A reports the expected time between trades in the stock. Panel B reports the illiquidity discount in the stock price as a percentage of $\bar{P}$, the price itself. Panel C reports the "illiquidity return premium" in the stock's rate of return (defined as the increase in the expected rate of return on the stock for positive transactions costs). Panel D reports the annual turnover ratio of the stock. And panel E reports the fixed transactions cost as a fraction of the average trade size, given by $\delta \times \bar{P}$.

From table 1, we observe that for a given level of risk aversion γ and the variability of nontraded risk σ_X, the time between trades, the illiquidity price discount, and the illiquidity return premium all increase with the transactions cost, and the average turnover decreases with the transactions cost. For example, for $\gamma = 2.5$ and $\sigma_X = 1.0$, the average time between trades increases from 0.147 year (seven trades per year) to 1.049 years (one trade per year) when the transactions cost increases from 0.1 percent of the share price to 5.0 percent of the share price. For the same increase in the transactions cost, the illiquidity discount

[10] The purpose of our calibration exercise is to develop a sense for the magnitude of the impact of transactions costs on prices and trading volume, not to validate the particular model considered here. Thus we have omitted the details of our calibration, which can be found in our working paper (Lo et al. 2001).

TABLE 1
CALIBRATION RESULTS ON THE PRICE IMPACT OF TRANSACTIONS COSTS FOR DIFFERENT VALUES OF THE RISK AVERSION γ AND TRANSACTIONS COST κ

$\kappa/\bar{P}$	$\gamma=.5$ FOR $\sigma_X=$			$\gamma=1.0$ FOR $\sigma_X=$			$\gamma=2.5$ FOR $\sigma_X=$		
(%)	.2	1.0	5.0	.2	1.0	5.0	.2	1.0	5.0
	A. $\Delta\tau$ (Years)								
.1	3.086	.612	.122	1.971	.392	.078	.738	.147	.030
.5	6.997	1.372	.273	4.448	.878	.175	1.656	.330	.066
1.0	9.999	1.944	.387	6.332	1.244	.248	2.348	.467	.094
5.0	23.362	4.385	.866	14.564	2.797	.555	5.309	1.049	.214
	B. Illiquidity Discount (Percentage of $\bar{P}$)								
.1	.003	.017	.083	.021	.104	.523	.345	1.733	8.846
.5	.007	.037	.187	.046	.233	1.187	.774	3.893	20.668
1.0	.011	.053	.266	.066	.330	1.702	1.095	5.537	30.348
5.0	.024	.119	.601	.147	.742	4.080	2.447	12.618	75.254
	C. Return Premium (%)								
.1	.000	.001	.004	.001	.006	.028	.052	.264	1.453
.5	.000	.002	.008	.002	.012	.064	.117	.606	3.901
1.0	.000	.002	.012	.003	.018	.092	.166	.878	6.525
5.0	.001	.005	.026	.008	.040	.225	.376	2.162	45.538
	D. Annual Turnover (%)								
.1	3.990	44.817	501.523	4.993	55.983	626.226	8.160	91.308	1,019.865
.5	2.650	29.929	335.291	3.324	37.404	418.655	5.447	61.019	680.702
1.0	2.217	25.141	281.884	2.786	31.431	351.967	4.575	51.289	571.556
5.0	1.450	16.739	188.334	1.837	20.959	235.146	3.042	34.220	378.949
	E. Cost as a Percentage of Transaction Amount								
.1	.081	.036	.016	.102	.046	.020	.167	.076	.036
.5	.270	.122	.055	.338	.153	.069	.559	.258	.140
1.0	.451	.205	.092	.567	.257	.117	.941	.442	.267
5.0	1.476	.682	.308	1.872	.859	.399	3.173	1.594	2.493

NOTE.—Other parameters are set at the following values: the interest rate $r = 0.0370$, annual dividend rate $\bar{a}_D = 0.05$, annual dividend volatility $\sigma_D = 0.2853$, and price level $\bar{P} = 0.7409$. Panel A reports expected trade interarrival times $\Delta\tau$ (in years), panel B reports the illiquidity discount in the stock price (as a percentage of the price $\bar{P}$ in the frictionless economy), panel C reports the return premium (defined as $[\bar{a}_D/P] - [\bar{a}_D/\bar{P}]$, where P is the price under the transactions cost), panel D reports the annual turnover in percentages, $100 \times (\delta/2\theta\tau)$, and panel E reports the transactions cost as a percentage of the transaction amount, $100 \times (\kappa/\delta P)$. These quantities are reported as functions of the transactions cost $\kappa_P \equiv \kappa/\bar{P}$ (in percentages) and the absolute risk aversion coefficient γ.

in the share price increases from 1.733 percent to 12.618 percent, and the illiquidity premium in the rate of return increases from 0.264 percent to 2.162 percent. The turnover, however, decreases from 91.308 percent to 34.220 percent per year.

The magnitude of the impact of transactions cost, however, depends on the value of σ_X, the amount of high-frequency trading needs. For example, for a transactions cost of 1.0 percent of the share price, the average time between trades is 2.348 years (one trade per two years) when $\sigma_X = 1.0$ versus 0.094 year (11 trades per year) when $\sigma_X = 5.0$, and the turnover is 4.575 percent versus 571.556 percent per year. More interesting, for the same two cases, the illiquidity price discount is 1.095 percent versus 30.348 percent, and the illiquidity return premium is 0.166 percent versus 6.525 percent. Clearly, for larger values of σ_X, agents have stronger motives for high-frequency trading, and the transactions cost has a larger impact on equilibrium prices and expected returns.

For a given value of σ_X and the transactions cost, the average time between trades decreases with risk aversion, and the illiquidity price discount, the illiquidity return premium, and the turnover all increase with risk aversion. For example, in table 1, for a transactions cost of 1.0 percent of the share price, the average time between trades ranges from 1.944 years (one trade per two years) to 0.467 year (two trades per year) when the value of the risk aversion coefficient γ increases from 0.5 to 2.5. For the same range of γ, the illiquidity price discount increases from 0.053 percent to 5.537 percent, the illiquidity return premium increases from 0.002 percent to 0.878 percent, and the turnover increases from 25.141 percent to 51.289 percent.

In choosing the values of the transactions cost in our calibration exercise, we have used the transactions cost as a fraction of the stock price. However, the level of the stock price we use is derived from the estimates of Campbell and Kyle for detrended prices; hence the interpretation of its magnitude is somewhat ambiguous. To better gauge the magnitude of the transactions cost as implied by our choice of fixed transactions cost, we report in panel E of table1 the cost κ as a percentage of the total transaction amount $\delta \times \bar{P}$, that is, $100 \times (\kappa/\delta\bar{P})$. This normalized measure of the transactions cost also depends on the choice of fixed cost, σ_X, and the risk aversion parameter. From table 1, for example, we see that it ranges from 0.081 to 2.493 percent of the total transaction amount, which seems quite plausible from an empirical perspective.

Table 1 shows that our model is capable of yielding realistic values for trading frequency, trading volume, and the illiquidity discount in the stock price, in contrast to much of the existing literature. For example, Schroeder (1998) finds that when faced with a fixed transactions cost of 0.1 percent of the total trade amount, an agent with a coefficient of relative risk aversion of 5.0 trades once every 10 years. In table 1, we see that for a fixed cost of approximately 0.1 percent or less of the total trade amount, agents in our model trade anywhere between once every 0.030 year (or 33 times a year) and once every 3.086 years. Even with relatively low levels of high-frequency trading needs, that is, when $\sigma_X = 1$, the turnover can range from 44.817 percent to 91.308 percent for different values of risk aversion and transactions cost. This is compatible with the average turnover in the U.S. stock market, which is 92.56 percent per year for the New York and American Stock Exchanges from 1962 to 1998 (see, e.g., Lo and Wang 2000).

Our calibration exercise shows that small transactions costs can have significant implications for equilibrium asset prices. For example, in table 1, for a modest value of 1.0 for σ_X, a transactions cost of 1 percent of the share price can give rise to a 5.337 percent discount in the stock price and an increase of 0.878 percent in expected returns when the risk aversion coefficient is 2.5. If the transactions cost becomes 5 percent of the share price (which is only 1.594 percent of the average transaction amount), the price discount due to illiquidity becomes 12.618 percent and the return premium becomes 2.162 percent, which are quite significant. When $\sigma_X = 5.0$, the illiquidity discount reaches 75.254 percent and the return premium becomes 45.538 percent. The significant impact of a small transactions cost in our model is in sharp contrast to the results in Constantinides (1986), Heaton and Lucas (1996), and Vayanos (1998).

The striking difference between our results and those of the existing literature stems from the fact that agents in our model have a strong desire to trade frequently—not trading can be very costly. Most of the other transactions cost models fail to capture high-frequency trading needs.[11] In table 2, we compare the impact of transactions cost on the stock price and return when the agents have high-frequency (i.e., stochastic) trading needs versus when they have only low-frequency (i.e., deterministic) trading needs. The latter case is discussed in Section V*D*. We have chosen the parameter for deterministic trading needs, a_X, to be three so that the resulting trading frequency and volume are comparable to the case with stochastic trading needs. It is apparent that with deterministic trading needs, the transactions cost has a negligible impact on a stock's price and return. This is in sharp contrast to the significant price impact that transactions costs can have when trading needs are stochastic.

Our results provide compelling motivation for focusing on high-frequency trading needs in any model of transactions costs in asset markets.

VII. Conclusions

We have developed a continuous-time equilibrium model of asset prices and trading volume with heterogeneous agents and fixed transactions costs. With prices, trading volume, and interarrival times determined endogenously, we show that even a small fixed cost of trading can have a substantial impact on the frequency of trade. Investors follow an optimal policy of not trading until their risk exposure reaches either a lower or upper boundary, at which point they incur the fixed cost and

[11] While partial equilibrium models such as those in Amihud and Mendelson (1986*b*) and Constantinides (1986) do contain a high-frequency component in agents' trading needs, they do not take into account the unwillingness of agents to hold the market-clearing level of the risky asset in the presence of transactions costs.

TABLE 2
Comparison between the Price Impact of Transactions Costs When Agents' Trading Needs Are Deterministic and When They Are Stochastic

$\kappa/\bar{P}$ (%)	$\gamma=.5$ Deterministic	$\gamma=.5$ Stochastic	$\gamma=1.0$ Deterministic	$\gamma=1.0$ Stochastic	$\gamma=2.5$ Deterministic	$\gamma=2.5$ Stochastic
	A. $\Delta\tau$ (Years)					
.1	1.212	.612	.902	.392	.470	.147
.5	2.073	1.372	1.542	.878	.803	.330
1.0	2.612	1.944	1.942	1.244	1.012	.467
5.0	4.466	4.385	3.321	2.797	1.730	1.049
	B. Illiquidity Discount (Percentage of $\bar{P}$)					
.1	.000	.017	.000	.104	.000	1.733
.5	.000	.037	.000	.233	.000	3.893
1.0	.000	.053	.000	.330	.000	5.537
5.0	.000	.119	.000	.742	.000	12.618
	C. Return Premium (%)					
.1	.000	.001	.000	.006	.000	.264
.5	.000	.002	.000	.012	.000	.606
1.0	.000	.002	.000	.018	.000	.878
5.0	.000	.005	.000	.040	.000	2.162
	D. Annual Turnover (%)					
.1	30.000	44.817	30.000	55.983	30.000	91.308
.5	30.000	29.929	30.000	37.404	30.000	61.019
1.0	30.000	25.141	30.000	31.431	30.000	51.289
5.0	30.000	16.739	30.000	20.959	30.000	34.220

Note.—We consider different values of the risk aversion γ and transactions cost κ, with $\sigma_X = 1.0$. Other parameters are set at the following values: the interest rate $r = 0.0370$, annual dividend rate $\bar{a}_D = 0.05$, annual dividend volatility $\sigma_D = 0.2853$, and price level $\bar{P} = 0.7409$. Panel A compares the expected trade interarrival times $\Delta\tau$ (in years) in the two cases, panel B compares the illiquidity discount in the stock price (as a percentage of the price $\bar{P}$ in the frictionless economy), panel C compares the return premium (defined as $[\bar{a}_D/P] - [\bar{a}_D/\bar{P}]$, where P is the price under the transactions cost), and panel D compares the annual turnover in percentages, $100 \times (\delta/2\theta\tau)$. These quantities are reported as functions of the transactions cost $\kappa/\bar{P}$ (in percentages) and the absolute risk aversion coefficient γ.

trade back to an optimal level of risk exposure. As the agents' uncertainty in trading needs increases, their "no-trade" region increases as well, despite the fact that the expected time between trades declines. Agents optimally balance their desire to manage their overall risk exposure against the fixed cost of transacting.

We also show that small fixed costs can induce a relatively large premium in asset prices. The magnitude of this liquidity premium is more sensitive to the risk aversion of agents than the risk premium is. Because agents must incur a transactions cost with every trade, they do not rebalance very often. In between trades, they face some uncertainty as to the level of their holdings of the risky asset. This increases the effective risk faced by the agent for holding the risky asset, which reduces his demand for the risky asset at any given price. To clear the market, the equilibrium price must compensate agents for the illiquidity of the shares that they hold. The price effect, then, relies heavily on the market-clearing motive; hence partial equilibrium models are likely to underestimate the effect of transactions costs on asset returns because they ignore this mechanism. Our model also leads to interesting predictions about the actual trading process, including trade sizes and trading frequency. In particular, we establish a power law between trade size and trading frequency. In a separate paper (Lo et al. 2003), we test this relation empirically using transactions data around stock splits when transactions costs change, and our findings are remarkably consistent with the power law predicted by the theory.

Our model also serves as a bridge between the market microstructure literature and the broader equilibrium asset-pricing literature. In particular, despite the many market microstructure studies that relate trading behavior to market-making activities—the price discovery mechanism and trading costs (see, e.g., Bagehot 1971; Glosten and Milgrom 1985; Kyle 1985; Easley and O'Hara 1987; Grossman and Miller 1988; Wang 1994)—the connection between these micro-aspects of the trading process and how assets are priced in equilibrium has received relatively little attention. Our model is an attempt to provide a concrete link between the two. Moreover, our framework yields significant implications for the dynamics of order flow, the evolution of bid/ask spreads and depths, and other aspects of market microstructure dynamics. In particular, our model endogenizes not only the price at which trades are consummated but also the *times* at which trades occur. This feature distinguishes our model from existing models of trading behavior in the market microstructure literature, models in which order flow is almost always specified exogenously (e.g., Glosten and Milgrom 1985; Kyle 1985). A detailed analysis of the behavior of bid-ask prices, market depth, and the trading process in our model can be found in Lo et al. (2001).

Although our model has many interesting theoretical and empirical implications, it is admittedly a rather simple parameterization of a considerably more complex set of phenomena. In particular, our assumption of perfect correlation between the dividend and endowment flows is likely to exaggerate the hedging motive in our economy. If a perfect hedging vehicle were not available, then agents may trade less often. The persistence of the endowment shocks in our economy may increase both the illiquidity discount and the desire to trade. Moreover, we do not allow for an aggregate endowment component (indeed our aggregate endowment is exactly zero), which certainly does exist in reality. All of these are interesting and important extensions of our model to be explored in future research.

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