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A computational view of market efficiency

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We study market efficiency from a computational viewpoint. Borrowing from theoretical computer science, we define a market to be *efficient with respect to resources S* (e.g., time, memory) if no strategy using resources S can make a profit. As a first step, we consider memory-m strategies whose action at time t depends only on the m previous observations at times $t-m, \ldots, t-1$. We introduce and study a simple model of market evolution, where strategies impact the market by their decision to buy or sell. We show that the effect of optimal strategies using memory m can lead to 'market conditions' that were not present initially, such as (1) market spikes and (2) the possibility for a strategy using memory $m' > m$ to make a bigger profit than was initially possible. We suggest ours as a framework to rationalize the technological arms race of quantitative trading firms.

Keywords: Agent based modelling; Bound rationality; Complexity in finance; Behavioral finance

1. Introduction

Market efficiency—the idea that "prices fully reflect all available information"—is one of the most important concepts in economics. A large number of articles have been devoted to its formulation, statistical implementation, and refutation since Fama (1965a,b, 1970) and Samuelson (1965) first argued that price changes must be unforecastable if they fully incorporate the information and expectations of all market participants. The more efficient the market, the more random the sequence of price changes generated by it, and the most efficient market of all is one in which price changes are completely random and unpredictable.

According to the proponents of market efficiency, this randomness is a direct result of many active market participants attempting to profit from their information. Driven by profit opportunities, legions of investors pounce on even the smallest informational advantages at their disposal, and in doing so, they incorporate their information into market prices and quickly eliminate the profit opportunities that first motivated their trades. If this occurs instantaneously, as in an idealized world of 'frictionless' markets and costless trading, then prices fully reflect all available information. Therefore, no profits can be garnered from information-based trading because such profits must have already been captured. In mathematical terms, prices follow martingales.

This stands in sharp contrast to finance practitioners who attempt to forecast future prices based on past prices, and, perhaps surprisingly, some of them do appear to make consistent profits that cannot be attributed to chance alone.

1.1. Our contribution

In this paper we suggest that a reinterpretation of market efficiency in computational terms might be the key to reconciling this theory with the possibility of making profits based on past prices alone. We believe that it does not make sense to talk about market efficiency without taking into account that market participants have bounded resources. In other words, instead of saying that a market is 'efficient' we should say, borrowing from theoretical computer science, that a market is *efficient with respect to resources S*, e.g. time, memory, etc., *if no*

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strategy using resources S can generate a substantial profit. Similarly, we cannot say that investors act optimally given all the available information, but rather they act optimally within their resources. This allows for markets to be efficient for some investors, but not for others; for example, a computationally powerful hedge fund may extract profits from a market that looks very efficient from the point of view of a day-trader who has less resources at his disposal—arguably the *status quo*.

As is well-known, suggestions in this same spirit have already been made in the literature. For example, Simon (1955) argued that agents are not rational but *boundedly rational*, which can be interpreted in modern terms as bounded in computational resources. Many other works in this direction are discussed in section 1.2. The main difference between this line of research and ours is: while it appears that most previous works use sophisticated continuous-time models, in particular explicitly addressing the market-making mechanism (which sets the price given agents' actions), our model is simple, discrete, and abstracts from the market-making mechanism.

1.1.1. Our model and results. We consider an economy where there exists only one good. The daily returns (price differences) of this good follow a pattern, e.g.

$$(-1, 1, -1, 1, -1, 1, 2, -1, 1, -1, 1, \\ -1, 1, 2, -1, 1, -1, 1, -1, 1, 2, \ldots),$$

which we write as $[-1, 1, -1, 1, -1, 1, 2]$ (see figure 1). A good whose returns follow a pattern may arise from a variety of causes ranging from biological to political; think of seasonal cycles for commodities, or the 4-year presidential cycle. Although one can consider more complicated dependencies, working with these finite patterns keeps the mathematics simple and is sufficient for the points made below.

Now we consider a new agent A who is allowed to trade the good daily, with no transaction costs. Intuitively, agent A can profit whenever she can predict the sign of the next return. However, the key point is that A is computationally bounded: her prediction can only depend on the previous m returns. Continuing the above example and setting memory $m = 2$, we see that upon seeing returns $(-1, 1)$, A's best strategy is to guess that the next return will be negative: this will be correct for two out of the three occurrences of $(-1, 1)$ (in each occurrence of the pattern).

We now consider a model of market evolution where A's strategy impacts the market by pushing the return closer to 0 whenever it correctly forecasts the sign of the return—thereby exploiting the existing possibility of profit—or pushing it away from 0 when the forecast is incorrect—thereby creating a new possibility for profit. In other words, the evolved market is the difference between the original market and the strategy guess (we think of the guess in $\{-1, 0, 1\}$). For example, figure 2 shows how the most profitable strategy using memory $m = 2$ evolves the market given by the pattern $[-2, 2, -2, 2, -2, 2, 3]$ (similar to the previous example).

The main question addressed in this work is: *what do markets evolved by memory-bounded strategies look like?*

We show that the effect of optimal strategies using memory m can lead to 'market conditions' that were not present initially, such as (1) market spikes and (2) the possibility that a strategy using memory $m' > m$ can make

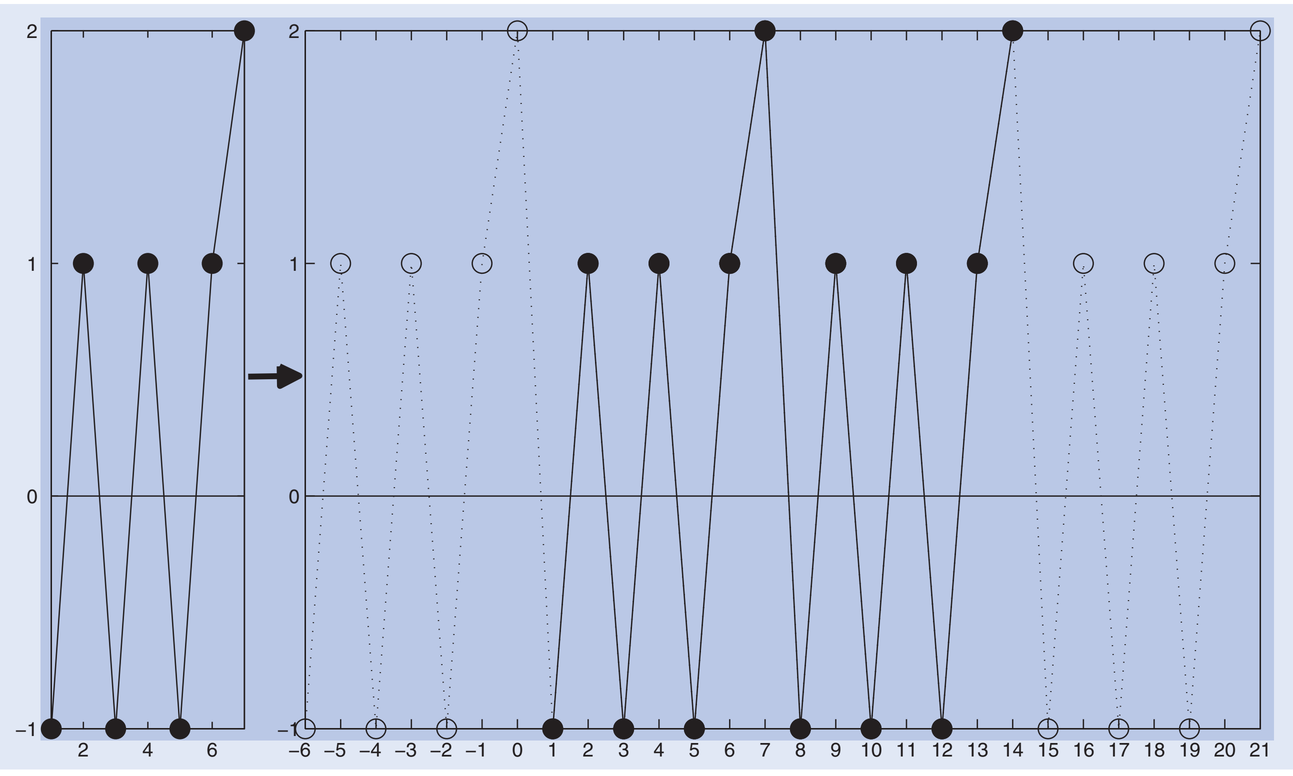

Figure 1. The market corresponding to a market pattern.

larger profits than previously possible. By market spikes (1), we mean that some returns will grow much larger, an effect already anticipated in figure 2. For point (2), we consider a new agent B that has memory m' that is larger than that of A. We let B trade on the market evolved by A. We show that, for some initial market, B may make more profit than what would have been possible if A had not evolved the market, or even if another agent with large memory m' had evolved the market instead of A. Thus, it is precisely the presence of low-memory agents (A) that allows high-memory agents (B) to make large profits.

Regarding the framework for (2), we stress that we only consider agents trading sequentially, not simultaneously: after A evolves the market, it becomes incorporated into a new market, on which B may trade. While a natural direction is extending our model to simultaneous agents, we argue that this sequentiality is not unrealistic. Nowadays, some popular strategies A are updated only once a month. Within any such month, a higher-frequency strategy B can indeed trade in the market evolved by A without A making any adjustment.

1.2. Related work

Since Fama's (1965a,b, 1970) and Samuelson's (1965) landmark papers, many others have extended their original framework, yielding a 'neoclassical' version of the efficient market hypothesis where price changes, properly weighted by aggregate marginal utilities, are unforecastable (see, for example, LeRoy (1973), Rubinstein (1976) and Lucas (1978)). In markets where, according to Lucas (1978), all investors have 'rational expectations', prices do fully reflect all available information and marginal-utility-weighted prices follow martingales. Market efficiency has been extended in many other directions, but the general thrust is the same: individual investors form expectations rationally, markets aggregate information efficiently, and equilibrium prices incorporate all available information instantaneously. See Lo (1997, 2007) for a more detailed summary of the market efficiency literature in economics and finance.

There are two branches of the market efficiency literature that are particularly relevant for our paper: the asymmetric information literature, and the literature on asset bubbles and crashes. In Black's (1986) presidential address to the American Finance Association, he argued that financial market prices were subject to 'noise', which could temporarily create inefficiencies that would ultimately be eliminated through intelligent investors competing against each other to generate profitable trades. Since then, many authors have modeled financial markets by hypothesizing two types of traders—informed and uninformed—where informed traders have private information regarding the true economic value of a security, and uninformed traders have no information at all, but merely trade for liquidity needs (see, for example, Grossman and Stiglitz (1980), Diamond and Verrecchia (1981), Admati (1985), Kyle (1985) and Campbell and Kyle (1993)). In this context, Grossman and Stiglitz (1980) argue that market efficiency is impossible because if markets were truly efficient, there would be no incentive for investors to gather private information and trade. DeLong *et al.* (1990, 1991) provide a more detailed analysis in which certain types of uninformed traders can destabilize market prices for periods of time even in the presence of informed traders. More recently, studies

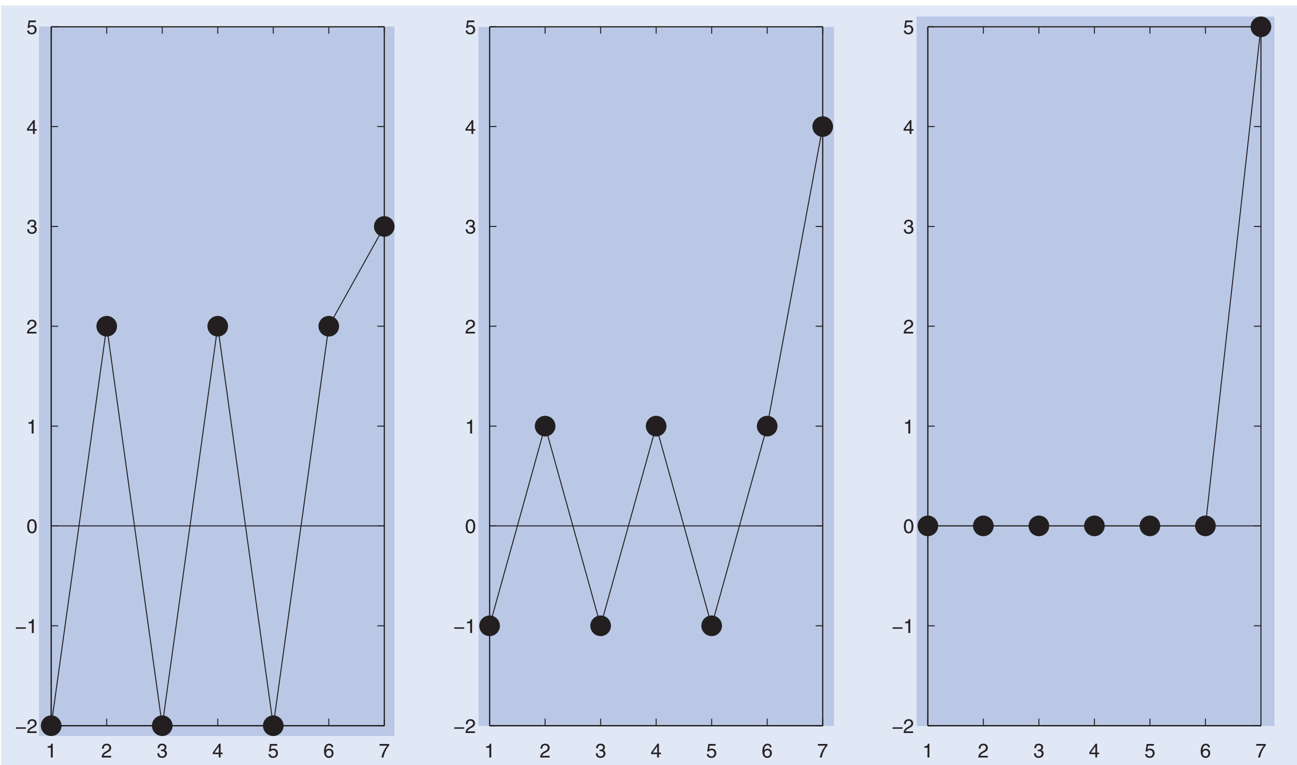

Figure 2. Optimal memory-2 strategy evolves the market and creates a spike.

by Luo (1995, 1998, 2001, 2003), Hirshleifer and Luo (2001) and Kogan *et al.* (2006) have focused on the long-term viability of noise traders when competing for survival against informed traders; while noise traders are exploited by informed traders as expected, certain conditions do allow them to persist, at least in limited numbers.

The second relevant offshoot of the market efficiency literature involves 'rational bubbles', in which financial asset prices can become arbitrarily large even when all agents are acting rationally, similarly to our market spikes (see, for example, Blanchard and Watson (1982), Allen *et al.* (1993), Santos and Woodford (1997) and Abreu and Brunnermeier (2003)). LeRoy (2004) provides a comprehensive review of this literature.

In the computer science literature, market efficiency has yet to be addressed explicitly. However, a number of studies have touched upon this concept tangentially. For example, computational learning methods have been applied to the pricing and trading of financial securities by Hutchinson *et al.* (1994). Evolutionary approaches in the dynamical systems and complexity literature have also been applied to financial markets by Farmer and Lo (1999), Farmer (2002), Farmer and Joshi (2002), Lo (2004, 2005), Farmer *et al.* (2005) and Bouchaud *et al.* (2008). And simulations of market dynamics using autonomous agents have coalesced into a distinct literature on 'agent-based models' of financial markets by Arthur (1994), Arthur *et al.* (1994, 1997, 1999), Farmer (2001), LeBaron (2001a–c, 2002, 2006) and Challet *et al.* (2004). In fact, our model has features that are similar to the 'El Farol' model of Arthur (1994) and the 'minority game' model (Challet *et al.* 2004). However, the mathematical understanding of those models requires familiarity with sophisticated techniques of statistical physics, while, by contrast, our techniques are elementary. Finally, we note that none of the papers in these distinct strands of the computational markets literature focus on a new definition of market efficiency, nevertheless they all touch upon different characteristics of this aspect of financial markets.

1.2.1. Organization.

We provide our computational definition of market efficiency in section 2. We describe the dynamics of market evolution in section 3, and conclude in section 4.

2. A computational definition of market efficiency

We model a market by an infinite sequence of random variables $X_1, X_2, \ldots,$ where $X_t \in \mathbb{R}$ is the *return* at time t. For the points made in this work, it is sufficient to consider markets that are obtained by the repetition of a pattern. This has the benefit of keeping the mathematics to a minimum level of sophistication.

Definition 2.1 (Market)**:** A *market pattern* of length p is a sequence of p random variables $[X_1, \ldots, X_p]$, where each $X_i \in \mathbb{R}$.

A market pattern $[X_1, \ldots, X_p]$ gives rise to a market obtained by the *independent* repetition of the patterns $X_1^1, \ldots, X_p^1, X_1^2, \ldots, X_p^2, X_1^3, \ldots, X_p^3, \ldots,$ where each block of random variables $X_1^i, \ldots, X_p^i$ is distributed as $X_1, \ldots, X_p$ and independent of $X_1^j, \ldots, X_p^j$ for $j \neq i$. Figure 1 shows an example for a market pattern in which the random variables are constant.

We now define strategies. A memory-m strategy takes as input the previous m observations of market returns, and outputs 1 if it believes that the next return will be positive. We can think of this '1' as corresponding to a buy-and-sell order (which is profitable if the next return is indeed positive). Similarly, a strategy output of -1 corresponds to a sell-and-buy order, while 0 corresponds to no order.

Definition 2.2: A *memory-m strategy* is a map $s: \mathbb{R}^m \to \{-1, 0, 1\}$.

We now define the gain of a strategy over a market. At every market observation, the gain of the strategy is the sign of the product of the strategy output and the market return: the strategy gains a profit when it correctly predicts whether the next return will be positive or negative, and loses a profit otherwise. A more sophisticated definition could take into account the magnitude of the return, but for simplicity we only consider its sign. Since we think of markets as defined by the repetition of a pattern, it is sufficient to define the gain of the strategy over this pattern.

Definition 2.3 (Gain of a strategy)**:** The *gain* of a memory-m strategy $s: \mathbb{R}^m \to \{-1, 0, 1\}$ over market pattern $[X_1, \ldots, X_p]$ is

$$\sum_{i=1}^{p} E_{X_1,\ldots,X_p}[\mathrm{sign}(s(X_{i-m}, X_{i-m+1}, \ldots, X_{i-1}) \cdot X_i)],$$

where, for any k, the random variable $(X_{1-k\cdot p}, \ldots, X_{p-k\cdot p})$ is independent of all the others and distributed as $(X_1, \ldots, X_p)$. A memory-m strategy s is *optimal* over a market pattern if no memory-m strategy has a larger gain than s over that market pattern.

The gain of an optimal memory-1 strategy over a certain market is a concept that is intuitively related to the (first-order) *autocorrelation* of the market, i.e. the correlation between the return at time t and that at time $t+1$ (for random t), a well-studied measure of market efficiency (see Lo (2005, figure 2)). This analogy can be made exact up to a normalization for markets that are given by balanced sequences of $+1$, -1. In this case, higher memory can be thought of as an extension of autocorrelation. It may be interesting to apply this measure to actual data.

Given a deterministic market pattern (which is just a sequence of numbers such as $[2, -1, 5, \ldots]$), optimal strategies can easily be computed: it is easy to see that an optimal strategy outputs $+1$ on input $x \in \mathbb{R}^m$ if more than half the occurrences of x in the market pattern are followed by a positive value. We are now ready to give our definition of market efficiency.

Definition 2.4 (Market efficiency)**:** A market pattern $[X_1, \ldots, X_p]$ is *efficient* with respect to memory-m

strategies if no such strategy has strictly positive gain over $[X_1, \ldots, X_p]$.

For example, let X_1 be a $\{-1, 1\}$ random variable that is -1 with probability 1/2. Then the market pattern $[X_1]$ is efficient with respect to memory-m strategies for every m. A standard 'parity' argument gives the following hierarchy, which we prove for the sake of completeness.

Claim 2.5: For every m, there is a market pattern that is efficient for memory-m strategies, but is not efficient for memory-$(m+1)$ strategies.

Proof: Let $X_1, X_2, \ldots, X_{m+1}$ be i.i.d. random variables with range $\{-1, 1\}$ whose probability of being 1 is 1/2. Consider the market pattern

$$\left[X_1, X_2, \ldots, X_{m+1}, X_{m+2} := \prod_{i=1}^{m+1} X_i\right]$$

of length $p = m+2$. By the definition of the pattern, the distribution of each variable is independent of the previous m, and therefore any memory-m strategy has gain 0. Consider the memory-$(m+1)$ strategy $s(x_1, x_2, \ldots, x_{m+1}) := \prod_{i=1}^{m+1} x_i$. Its gain over the market pattern is

$$\sum_{i=1}^{p} E_{X_1,\ldots,X_p}[\text{sign}(X_{i-m-1} \cdot X_{i-m} \cdots X_{i-1} \cdot X_i)]$$
$$= E_{X_1,\ldots,X_p}[\text{sign}(X_1 \cdot X_2 \cdots X_{m+1} \cdot X_{m+2})] = 1.$$

□

3. Market evolution

In this section we describe the dynamics of our market model. We consider a simple model of evolution where the strategies enter the market sequentially. After a strategy enters, its 'impact on the market' is recorded in the market and produces a new evolved market. The way in which a strategy impacts the market is by subtracting the strategy output from the market data.

Definition 3.1 (Market evolution)**:** We say that a memory-m strategy $s : \mathbb{R}^m \to \{-1, 0, 1\}$ *evolves* a market pattern $[X_1, X_2, \ldots, X_p]$ into the market pattern

$$[X_1 - s(X_{1-m}, \ldots, X_{1-1}), X_2 - s(X_{2-m}, \ldots, X_{2-1}), \ldots, X_p - s(X_{p-m}, \ldots, X_{p-1})].$$

Figure 2 shows the evolution of a market pattern. Definition 3.1 can readily be extended to multiple strategies acting simultaneously, just by subtracting all the impacts of the strategies on the market, but for the points made in this paper the above one is sufficient. In the next two subsections we point out two consequences of the above definition.

3.1. Market spikes

In this section we point out how low-memory strategies can give rise to *market spikes*, i.e. we show that an optimal low-memory strategy can evolve a market pattern into another one which has values that are much bigger than those of the original market pattern (in absolute value). Consider, for example, the market pattern $[-2, 2, -2, 2, -2, 2, 3]$ in figure 2. Note how an *optimal* memory-2 strategy will profit at most time instances, but not all. In particular, on input $(-2, 2)$, the best output is -1, which agrees with the sign of the market in two out of three occurrences of $(-2, 2)$. This shrinks the returns towards 0 for most time instances, but will make the last return in the pattern rise. The optimal memory-2 strategy evolves the original market pattern into $[-1, 1, -1, 1, -1, 1, 4]$. The situation then repeats, and an optimal memory-2 strategy evolves the latter pattern into $[0, 0, 0, 0, 0, 0, 5]$. This is an example of how an optimal strategy creates market conditions that were not initially present. We point out that the market spike does not form with memory 3 (see figure 3).

We also point out that the formation of such spikes is not an isolated phenomenon, nor specific to memory 2: we have generated several random markets and plotted their evolutions, and spikes often arise with various memories. We report one such example in figure 4, where a random pattern of length 20 is evolved by optimal memory-3 strategies into a spike after 16 iterations. In fact, the framework proposed in this paper gives rise to a 'game' in the spirit of Conway's Game of Life, showing how simple trading rules can lead to apparently chaotic market dynamics. Perhaps the main difference between our game and Conway's is that, in ours, strategies perform optimally within their resources.

3.2. High-memory strategies feed off low-memory strategies

The next claim shows a pattern where a high-memory strategy can make a bigger profit after a low-memory strategy has acted and modified the market pattern. This profit is bigger than the profit that is obtainable by a high-memory strategy both in the case in which no strategy acts beforehand and in the case in which another high-memory strategy acts beforehand. Thus it is precisely the presence of low-memory strategies that creates opportunities for high-memory strategies which were not present initially. This example provides an explanation for the real-life *status quo* which sees a growing quantitative sophistication among asset managers.

Informally, the proof of the claim exhibits a market with a certain 'symmetry'. For high-memory strategies, the best choice is to maintain the symmetry by profiting in multiple points. But a low-memory strategy will be unable to do so. Its optimal choice will be to 'break the symmetry', creating new profit opportunities for high-memory strategies.

Claim 3.2: For every $m < m'$ there is a market pattern $P = [X_1, \ldots, X_p]$, an optimal memory-m strategy s_m that evolves P into P_m, and an optimal memory-m' strategy $s_{m'}$ that evolves P into $P_{m'}$ such that the gain of an optimal

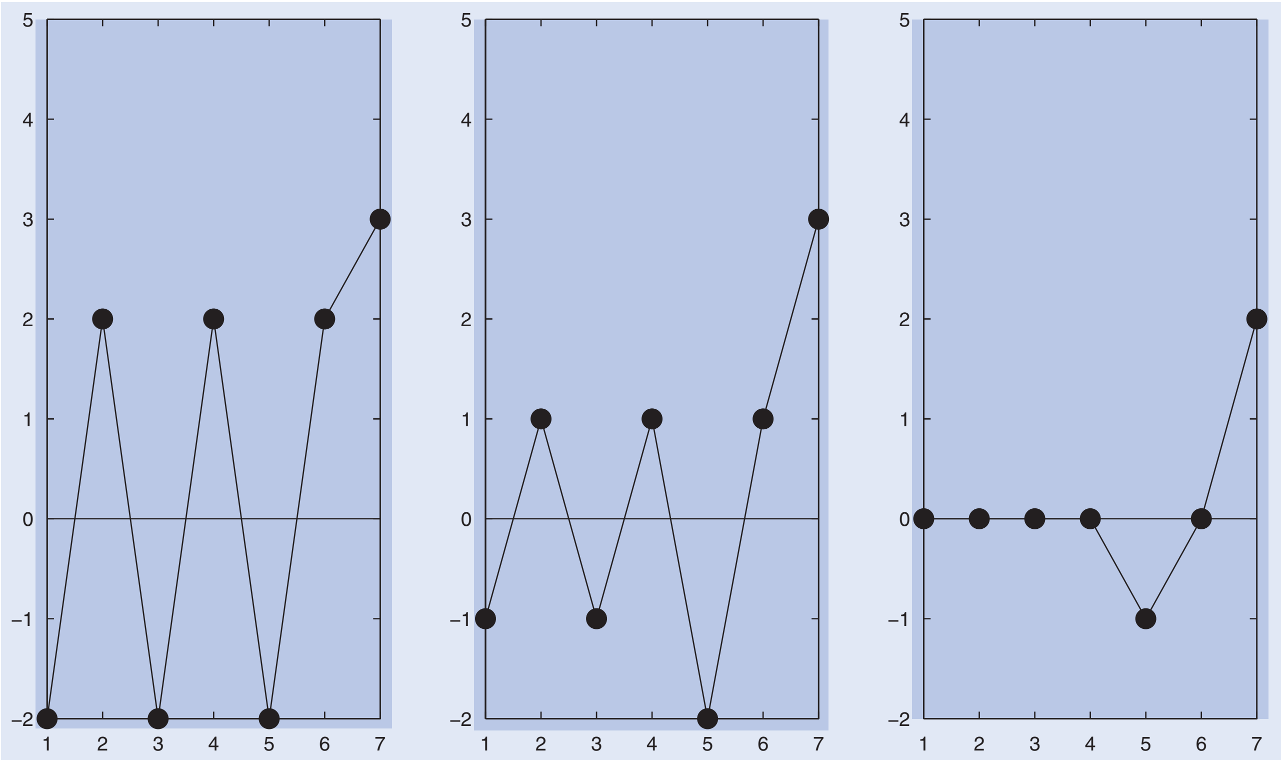

Figure 3. Optimal memory-3 strategy evolves the market without creating a spike.

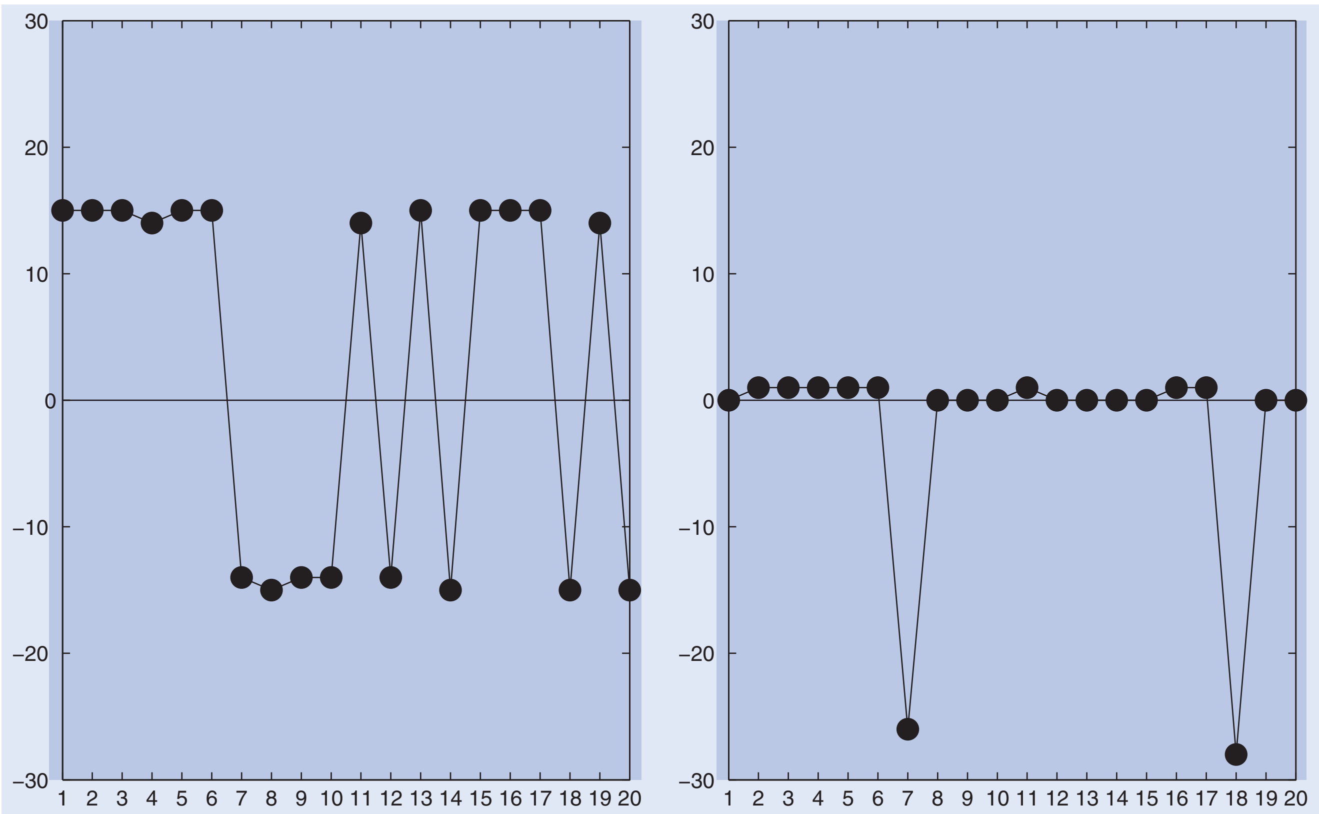

Figure 4. Random market pattern (left) evolved by an optimal memory-3 strategy into a spike after 16 iterations (right).

memory-m' strategy over P_m is bigger than either of the following:

- the gain of $s_{m'}$ over P;
- the gain of any memory-m strategy over P_m;
- the gain of any memory-m' strategy over $P_{m'}$.

Proof: Consider the market pattern

$$\left[X_1,\ldots,X_{m-1},a\cdot X_m,b\prod_{i=1}^{m}X_i,Y_2,\ldots,Y_{m'-1},\right.$$
$$c\cdot Y_{m'},\prod_{i=1}^{m}X_i\prod_{i=2}^{m'}Y_i,X_1,\ldots,X_{m'-1},a'\cdot X_{m'},$$
$$\left.b\prod_{i=1}^{m'}X_i,Y_2,\ldots,Y_{m'-1},c\cdot Y_{m'},-\prod_{i=1}^{m'}X_i\prod_{i=2}^{m'}Y_i\right],$$

where $X_1,\ldots,X_{m'}$ and $Y_2,\ldots,Y_{m'}$ are independent $\{-1,1\}$ random variables (note that some of these variables appear multiple times in the pattern), and $a=10$, $a'=20$, $b=30$, $c=40$. We now analyse the gains of various strategies. For this it is convenient to define the *latest input* of a strategy $s(x_1,\ldots,x_\ell)$ as x_ℓ; this corresponds to the most recent observation the strategy is taking into consideration.

The strategy s_m. We note that there is an optimal strategy s_m on P that outputs 0 unless its latest input has absolute value a. This is because, in all other instances, the random variable the strategy is trying to predict is independent of the previous m. Thus, $s_m(x_1,\ldots,x_m)$ equals 0 unless $|x_m|=a$, in which case it outputs the sign of $\prod_{i=1}^{m}x_i$. The gain of this strategy is 1. The strategy evolves the pattern into the same pattern except that the *first* occurrence of b is replaced by $b-1$:

$$P_m:=\left[X_1,\ldots,X_{m-1},a\cdot X_m,(b-1)\prod_{i=1}^{m}X_i,Y_2,\ldots,Y_{m'-1},\right.$$
$$c\cdot Y_{m'},\prod_{i=1}^{m}X_i\prod_{i=2}^{m'}Y_i,X_1,\ldots,X_{m'-1},a'\cdot X_{m'},$$
$$\left.b\prod_{i=1}^{m'}X_i,Y_2,\ldots,Y_{m'-1},c\cdot Y_{m'},-\prod_{i=1}^{m'}X_i\prod_{i=2}^{m'}Y_i\right].$$

The strategy $s_{m'}$. We note that there is an optimal memory-m' strategy over P that outputs 0 unless its latest input has absolute value a, a', or c. Again, this is because, in all other instances, the strategy is trying to predict a variable that is independent of the previous m'. Moreover, when its latest input has absolute value c, we can also assume that the strategy outputs 0. This is because the corresponding contribution is

$$E\left[\operatorname{sign}\left(s_{m'}\left(b\cdot\prod_{i=1}^{m}X_i,Y_2,\ldots,Y_{m'-1},c\cdot Y_{m'}\right)\cdot\prod_{i=1}^{m}X_i\prod_{i=2}^{m'}Y_i\right)\right]$$
$$+E\left[\operatorname{sign}\left(s_{m'}\left(b\cdot\prod_{i=1}^{m'}X_i,Y_2,\ldots,Y_{m'-1},c\cdot Y_{m'}\right)\right.\right.$$
$$\left.\left.\cdot-\prod_{i=1}^{m'}X_i\prod_{i=2}^{m'}Y_i\right)\right]=0.$$

Thus there is an optimal memory-m' strategy with gain 2 that evolves the market pattern into the pattern $P_{m'}$ that is like P with b replaced by $b-1$ in *both* occurrences (as opposed to P_m which has b replaced by $b-1$ only in the first occurrence).

The optimal memory-m' strategy on $P_{m'}$. Since $P_{m'}$ is like P with b replaced by $b-1$, the gain of the optimal memory-m' strategy on $P_{m'}$ is 2.

The optimal memory-m strategy on P_m. Essentially the same argument for s_m can be applied again to argue that any memory-m strategy on P_m has gain 1 at most.

The optimal memory-m' strategy on P_m. Finally, note that there is a memory-m' strategy whose gain is 4 on P_m. This is because the replacement of the first occurrence of b in P with $b-1$ allows a memory-m' strategy to predict the sign of the market correctly when its latest input has absolute value c. □

We make a few final comments regarding claim 3.2. First, the same example can be obtained for deterministic market patterns by considering the exponentially longer market pattern where the random variables take all possible combinations. However, the randomized example is easier to describe. Also, although in claim 3.2 we talk about *an* optimal strategy, in fact the particular strategies considered there are natural in that they are those that minimize the number of $\{-1,1\}$ outputs (i.e. the amount of trading). We could have forced this to be the case by adding to our definition of gain a negligible transaction cost, in which case the claim would talk about *the* optimal strategy, but we preferred a simpler definition of gain.

4. Conclusion

In this work we have suggested the study of market efficiency from a computational point of view. We have put forth a specific memory-based framework that is simple and tractable, yet capable of modeling market dynamics such as the formation of market spikes and the possibility for a high-memory strategy to 'feed off' low-memory strategies. Our results may provide an analytical framework for studying the technological arms race that portfolio managers have been engaged in since the advent of organized financial markets.

Our framework also gives rise to a few technical questions, such as how many evolutions does it take a pattern to reach a certain other pattern, to what extent does this number of evolutions depend on different levels of memory, and to what extent does it depend on the simultaneous interaction among strategies using different memories.

A natural next step is to consider different classes of strategies, with other types of computational restrictions. Here, the models studied by theoretical computer science offer ample choice.

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